Problem 5

The equation of the tangent line at the point $(x_0, y_0) = (2, -4)$ is

$$y + 4 = m(x - 2)$$

where $m = f'(2)$. The derivative is given by the limit of the different quotient:

$$\begin{aligned}\frac{f(2+h) - f(2)}{h} &= \frac{4(2+h) - 3(2+h)^2 + 4}{h} \\ &= \frac{8 + 4h - 3(4 + 4h + h^2) + 4}{h} \\ &= \frac{-4 - 8h - 3h^2 + 4}{h} \\ &= -8 - 3h\end{aligned}$$

and as $h \to 0$, we get $f'(2) = -8$. So, we get

$$y + 4 = -8(x - 2).$$

Problem 6

The equation of the tangent line at $(2, 3)$ is

$$y - 3 = f'(2)(x - 2).$$

We have to find $f'(2)$. We have $f(x) = x^3 - 3x + 1$, and therefore

$$\begin{aligned}f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0} \frac{(2+h)^3 - 3(2+h) + 1 - 3}{h} \\ &= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h + 1 - 3}{h} \\ &= \lim_{h \to 0} \frac{9h + 2h^2 + h^3}{h} \\ &= \lim_{h \to 0} 9 + 2h + h^2 \\ &= 9.\end{aligned}$$

Therefore, we obtain $f'(2) = 9$. Therefore, the equation of the tangent line is

$$y = 9x - 18 + 3 = 9x - 15.$$

Problem 34

The value of $f'(a)$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Evaluating f at $a+h$ and at a in this expression, we can do some calculations:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h} \\
&= \lim_{h \to 0} \frac{a^2 - (a+h)^2}{(a+h)^2 a^2 h} \\
&= \lim_{h \to 0} \frac{a^2 - a^2 - 2ah - h^2}{(a+h)^2 a^2 h} \\
&= \lim_{h \to 0} -\frac{2ah + h^2}{(a+h)^2 a^2 h} \\
&= \lim_{h \to 0} -\frac{2a + h}{(a+h)^2 a^2} \\
&= -\frac{2a}{a^4} \\
&= -\frac{2}{a^3}.
\end{aligned}$$

Therefore, we get $f'(a) = -2/a^3$.

Problem 44

The velocity at $t = 4$ is given by $f'(4)$. This is given by

$$\begin{aligned}
\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h \to 0} \frac{10 + \frac{45}{5+h} - 10 - \frac{45}{5}}{h} \\
&= \lim_{h \to 0} \frac{\frac{45}{5+h} - 9}{h} \\
&= \lim_{h \to 0} \frac{45 - 45 - 9h}{(5+h)h} \\
&= \lim_{h \to 0} -\frac{9h}{(5+h)h} \\
&= \lim_{h \to 0} -\frac{9}{5+h}.
\end{aligned}$$

Evaluating the last limit with the Quotient Rule, we get $f'(4) = -9/5$.

Problem 60

By definition, we have

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} \\ &= \lim_{h\to 0} h\sin(1/h). \end{aligned}$$

The last limit exists because

$$-h \le h\sin(1/h) \le h$$

for any $h > 0$ and

$$h \le h\sin(1/h) \le -h$$

when $h < 0$. We can simplify this by using the absolute value:

$$0 \le |h\sin(1/h)| \le |h|$$

because $0 \le |\sin(1/h)| \le 1$. Using the Squeeze Theorem, we conclude that

$$\lim_{h\to 0} h\sin(1/h) = 0.$$

Therefore, $f'(0)$ exists and $f'(0) = 0$.