Problem 12

Since f is a polynomial, then it is continuous and differentiable on $[-2, 2]$. Therefore, the hypothesis of the MVP are satisfied. We want to find all solutions c to

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{f(2) - f(-2)}{4}.$$

The derivative of f is

$$f'(x) = 3x^2 - 3.$$

Therefore, we look for numbers c such that

$$3c^2 - 3 = \frac{4 - 0}{4} = 1.$$

So, c should be a solution of

$$3c^2 = 4 \iff c = \pm\frac{2}{\sqrt{3}}.$$

The numbers that satisfy the Mean Value Theorem are $c = -2/\sqrt{3}$ and $c = 2/\sqrt{3}$.

Problem 20

The function $f(x) = 2x - 1 - \sin x$ is continuous. It is also differentiable at every point. We can apply the IVT and the MVT.

We first use the IVT to show that there is at least one root. We see that $f(0) = -1 < 0$ and $f(\pi) = 2\pi - 1 > 0$. So, letting $N = 0$ in the IVT, we conclude that there is a number c between 0 and π such that $f(c) = 0$.

We secondly use the MVT to show that there is only one root. The derivative of $f(x)$ is $f'(x) = 2 - \cos x$. If there were two roots to the equation $f(x) = 0$, call them c_1 and c_2, then $f(c_1) = f(c_2) = 0$ and from the MVT we conclude that there is a $\tilde{c}$ between c_1 and c_2 such that $f'(\tilde{c}) = 0$. But $f'(x) = 2 - \cos x > 0$ for any number x because $-1 \leq \cos x \leq 1$. This is a contradiction. So, there must be only one root to the equation $f(x) = 0$.

Problem 30

Fix $b > 0$. An odd function on $[-b, b]$ means that $f(-x) = -f(x)$ for any x in $[-b, b]$.

Since f is differentiable, from the Mean Value Theorem, there exists a c in $(-b, b)$ such that

$$f'(c) = \frac{f(b) - f(-b)}{2b}.$$

However, $f(-b) = -f(b)$ and therefore

$$\frac{f(b) - f(-b)}{2b} = \frac{f(b) + f(b)}{2b} = \frac{f(b)}{b}.$$

So, combining everything together, there exists a c in $(-b, b)$ such that

$$f'(c) = \frac{f(b)}{b}.$$

This completes the proof.