Problem 10

a) The derivative of f is

$$f'(x) = 6x^2 - 18x + 12.$$

We see that

$$f'(x) = 6(x - 3x + 2) = 6(x - 2)(x - 1).$$

Therefore, the zeros of f' are $x = 2$ and $x = 1$.

- if $x < 1$, then $x < 2$ also. Therefore, $x-1 < 0$ and $x-2 < 0$. The product $(x-2)(x-1)$ is positive, being the product of two negative quantities. So $f'(x) > 0$ when $x < 1$. Hence f is increasing for $x < 1$.
- if $x > 1$ and $x < 2$. Therefore, $x - 1 > 0$ and $x - 2 < 0$. The product $(x - 2)(x - 1)$ is negative, being the product of a negative quantity by a positive quantity. So $f'(x) < 0$ when $x > 1$ and $x < 2$. Hence f is decreasing for $1 < x < 2$.
- If $x > 2$, then $x > 1$ also. Therefore $x-1 > 0$ and $x-2 > 0$. The product $(x-2)(x-1)$ is positive, being the product of two positive quantities. So $f'(x) > 0$ when $x > 2$. Hence f is increasing for $x > 2$.

b) We know that $f'(x) = 6(x - 2)(x - 1)$. Therefore, the zeros of the derivative are $x = 1$ and $x = 2$. The derivative exists everywhere.

- $x = 1$. In this case, we see that f is increasing for $x < 1$ and f is decreasing for $1 < x < 2$. Therefore, from the first derivative test, f attains a local maximum at $x = 1$.
- $x = 2$. In this case, we see that f is decreasing for $1 < x < 2$ and increasing for $x > 2$. Therefore, from the first derivative test, f attains a local minimum at $x = 2$.

c) The second derivative of f is

$$f''(x) = 12x - 18 = 6(2x - 3).$$

The zeros of f'' are $x = 3/2$.

- When $x < 3/2$, then $2x - 3 < 0$. Therefore, $f''(x) < 0$. This means that f is concave downward.
- When $x > 3/2$, then $2x - 3 > 0$. Therefore, $f''(x) > 0$. This means that f is concave up..

Problem 12

(a) The derivative of f is

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1 - x^2}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}.$$

The critical points are at $x = 1$ and $x = -1$ where $f'(x)$ is zero. The derivative exists for any x.

When $x < -1$, then $(1 - x) > 0$ and $1 + x < 0$. The denominator is always positive and therefore $f'(x) < 0$. The function is decreasing when $x < -1$.

When $-1 < x < 1$, then $1 - x > 0$ and $1 + x > 0$. The denominator is always positive and therefore $f'(x) > 0$. The function is increasing for $-1 < x < 1$.

When $x > 1$, then $1-x < 0$ and $1+x > 0$. The denominator is always positive and therefore $f'(x) < 0$. The function is decreasing for $x > 1$.

(b) When $x < -1$, the function f is decreasing and when $-1 < x < 1$, the function f is increasing. By the first derivative test, $x = -1$ is a local minimum. The local minimum value of f there is therefore

$$f(-1) = -\frac{1}{2}.$$

When $-1 < x < 1$, the function f is increasing and when $x > 1$, the function f is decreasing. By the first derivative test, $x = 1$ is a local maximum. The local maximum value of f there is

$$f(1) = \frac{1}{2}.$$

(c) The second derivative is

$$\begin{aligned}
f''(x) &= \frac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4} \\
&= \frac{-2x(1+2x^2+x^4) - 4x(1-x^4)}{(1+x^2)^4} \\
&= \frac{-2x - 4x^3 - 4x^5 - 4x + 4x^5}{(1+x^2)^4} \\
&= \frac{-6x - 4x^3}{(1+x^2)^4} \\
&= -4\frac{x(3/2 - x^2)}{(1+x^2)^4}
\end{aligned}$$

and therefore

$$f''(x) = -4\frac{x(\sqrt{3/2} - x)(\sqrt{3/2} + x)}{(1+x^4)^4}.$$

The possible inflection points are when $f''(x) = 0$ or $f''(x)$ does not exist. There is no problem with the expression of $f''(x)$. The zeros are $x = -\sqrt{3/2}$, $x = 0$ and $x = \sqrt{3/2}$.

Here is a table summarizing all the information we need to answer the question. The detailed explanations are presented after the table.

Factors	$x <$	$-\sqrt{3/2}$	$< x <$	0	$< x <$	$\sqrt{3/2}$	$< x$
-4	$-$	$\cdot$	$-$	$\cdot$	$-$	$\cdot$	$-$
x	$-$	$\cdot$	$-$	$\cdot$	$+$	$\cdot$	$+$
$\sqrt{3/2} - x$	$+$	$\cdot$	$+$	$\cdot$	$+$	$\cdot$	$-$
$\sqrt{3/2} + x$	$-$	$\cdot$	$+$	$\cdot$	$+$	$\cdot$	$+$
$(1 + x^4)^4$	$+$	$\cdot$	$+$	$\cdot$	$+$	$\cdot$	$+$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\frown$	IP	$\smile$	IP	$\frown$	IP	$\smile$

When $x < -\sqrt{3/2}$, then $x < 0$, $\sqrt{3/2} - x > 0$, and $\sqrt{3/2} + x < 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) < 0$. Therefore, the function is concave down for $x < -\sqrt{3/2}$.

When $-\sqrt{3/2} < x < 0$, then $x < 0$, $\sqrt{3/2} - x > 0$ and $\sqrt{3/2} + x > 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) > 0$ there. Therefore, the function is concave up for $-\sqrt{3/2} < x < 0$.

Since f changes from concave down to concave up at $x = -\sqrt{3/2}$, the number $x = -\sqrt{3/2}$ is an inflection point.

When $0 < x < \sqrt{3/2}$, then $x > 0$, $\sqrt{3/2} - x > 0$, and $\sqrt{3/2} + x > 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) < 0$ there. Therefore, the function is concave down for $0 < x < \sqrt{3/2}$.

Since f changes from concave up to concave down at $x = 0$, the number $x = 0$ is an inflection point.

Finally, when $x > \sqrt{3/2}$, then $x > 0$, $\sqrt{3/2} - x < 0$, $\sqrt{3/2} + x > 0$. Since $-4 < 0$ and the denominator is always positive, we conclude that $f''(x) > 0$ there. Therefore, the function is concave up for $x > \sqrt{3/2}$.

Since f changes from concave down to concave up at $x = \sqrt{3/2}$, the number $x = \sqrt{3/2}$ is an inflection point.

Problem 16

With the First Derivative Test. The derivative of f is

$$f'(x) = \frac{x(x-2)}{(x-1)^2}.$$

The critical numbers of f are $c = 0$, $c = 1$, and $c = 2$.

- $c = 0$.
  - When $x < 0$, then $x - 2 < 0$. Since $(x - 1)$ is squared, then $(x - 1) > 0$. Therefore, we see that $f'(x) > 0$ for $x < 0$ because $f'(x)$ is the product of a two negative quantities and a positive quantity. So f is increasing when $x < 0$.
  - When $0 < x < 1$, then $x - 2 < 0$ and $x - 1 < 0$. But $(x - 1)$ is squared, so $(x - 1)^2 > 0$. Therefore, we see that $f'(x) < 0$ because $f'(x)$ is the product of two positive quantities and a negative quantity. Therefore, f is decreasing if $0 < x < 1$.

  Using the First Derivative Test, we conclude that $c = 0$ is a local maximum of f.

- $c = 1$.
  - When $0 < x < 1$, then $x - 2 < 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) < 0$ because it is a product of two positive quantities and a negative quantity. So f is decreasing on $0 < x < 1$.
  - When $1 < x < 2$, then $x - 2 < 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) < 0$ because it is a product of two positive quantities and a negative quantity. So f is decreasing on $1 < x < 2$.

  Since there is no change in the sign of $f'(x)$, we conclude that $c = 1$ is not a local maximum, nor a local minimum.

- $c = 2$.
  - When $1 < x < 2$, then $x - 2 < 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) < 0$ because it is a product of two positive quantities and a negative quantity. So f is decreasing on $1 < x < 2$.
  - When $x > 2$, then $x - 2 > 0$ and $x > 0$. Since $(x - 1)^2 > 0$, then $f'(x) > 0$ because it is the product of three positive quantities. So f is increasing on $x > 2$.

  By the First derivative Test, we conclude that f has a local minimum at $c = 2$.

With the Second Derivative Test. From the first part, we know that the critical numbers of f are $c = 0$, $c = 1$, and $c = 2$. The second derivative of f is

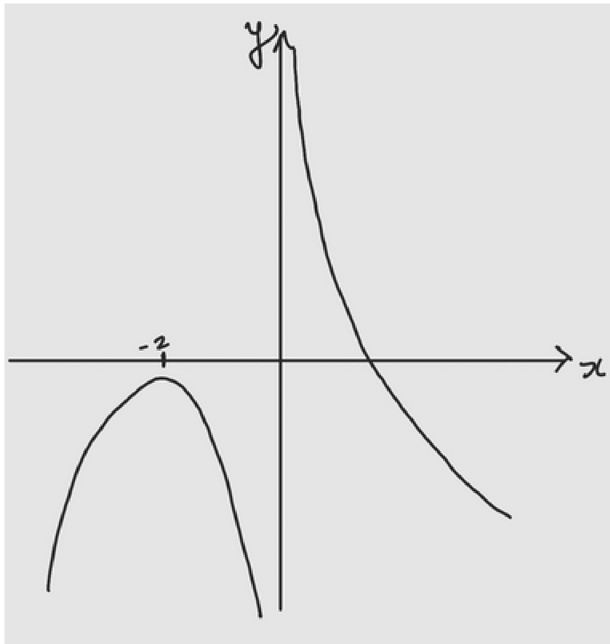
$$f''(x) = \frac{2}{(x-1)^3}.$$

- $c = 0$. We have $f''(0) = \frac{2}{(-1)^3} = -2$. Since $f''(0) < 0$, by the Second Derivative Test, we conclude that $c = 0$ is a local maximum.
- $c = 1$. We have $f''(1)$ does not exist. So we can't conclude anything from the Second Derivative Test.
- $c = 2$. We have $f''(2) = \frac{2}{(1)^3} = 2$. Since $f''(2) > 0$, by the Second Derivative Test, we conclude that $c = 2$ is a local minimum.

Remark: I personnaly prefer the Second Test Derivative over the First Derivative Test (but this is when we can use the second test).

Problem 22

Here is a possible graph of a function with the desire properties.



Problem 30

(a) The derivative and the second derivative are positive at B. The reasons are that, at B, the slope of the tangent line is positive and the graph of the function is concave up.

(b) The derivative and the second derivative are negative at E. The reasons for that are, at E, the slope of the tangent line is negative and the graph of the function is concave down.

(c) The derivative is negative and the second derivative is positive at A. The reasons for that are, at A, the slope of the tangent line is negative and the graph of the function is concave up.