Problem 8

We factor the greatest power of x:

$$\frac{9x^3+8x-4}{3-5x+x^3} = \frac{x^3(9+8/x^2-4/x^3)}{x^3(3/x^3-5/x^2+1)} = \frac{9+8/x^2-4/x^3}{3/x^3-5/x^2+1}.$$

We have

$$\lim_{x\to\infty} 9+8/x^2-4/x^3 = \lim_{x\to\infty} 9 + 8\lim_{x\to\infty} 1/x^2 - 4\lim_{x\to\infty} 1/x^3 = 9+8\times 0 - 4\times 0 = 9$$

and

$$\lim_{x\to\infty} 3/x^3-5/x^2+1 = 3\lim_{x\to\infty} 1/x^3 - 5\lim_{x\to\infty} 1/x^2 + \lim_{x\to\infty} 1 = 3\times 0 - 5\times 0 + 1 = 1.$$

So, we obtain

$$\lim_{x\to\infty} \frac{9x^3+8x-4}{3-5x+x^3} = \lim_{x\to\infty} \frac{9+8/x^2-4/x^3}{3/x^3-5/x^2+1} = \frac{\lim_{x\to\infty} 9+8/x^2-4/x^3}{\lim_{x\to\infty} 3/x^3-5/x^2+1} = \frac{9}{1} = 9$$

and then

$$\lim_{x\to\infty} \sqrt{\frac{9x^3+8x-4}{3-5x+x^3}} = \sqrt{\lim_{x\to\infty} \frac{9x^3+8x-4}{3-5x+x^3}} = \sqrt{9} = 3.$$

Problem 12

Factoring x^3, we have

$$\frac{4x^3+6x^2-2}{2x^3-4x+5} = \frac{4+6/x-2/x^3}{2-4/x^2+5/x^3}$$

Using the fact that $\lim_{x\to-\infty} \frac{1}{x^r} = 0$, we obtain

$$\lim_{x\to-\infty} \left(4+\frac{6}{x}-\frac{2}{x^3}\right) = 4 \quad \text{and} \quad \lim_{x\to-\infty} \left(2-\frac{4}{x^2}+\frac{5}{x^3}\right) = 2.$$

By the quotient rule, we see that

$$\lim_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} = \frac{4+6/x-2/x^3}{2-4/x^2+5/x^3} = \frac{4}{2} = 2.$$

Problem 14

Factoring $t^{3/2}$, we have

$$\frac{t-t^{3/2}}{2t^{3/2}+3t-5}=\frac{1/t^{1/2}-1}{2+3/t^{1/2}-5/t^{3/2}}.$$

Using the fact that $\lim_{t\to\infty}\frac{1}{t^r}=0$, we obtain

$$\lim_{t\to\infty}\left(\frac{1}{t^{1/2}}-1\right)=-1 \quad \text{and} \quad \lim_{t\to\infty}\left(2+\frac{3}{t^{1/2}}-\frac{5}{t^{3/2}}\right)=2.$$

By the quotient rule, we see that

$$\lim_{t\to\infty}\frac{t-t^{3/2}}{2t^{3/2}+3t-5}=\frac{-1}{2}.$$

Problem 16

Factoring x^4, we see that

$$\sqrt{x^4+1}=\sqrt{x^4}\sqrt{1+1/x^4}$$

Since $x\to\infty$, we must have that $x>0$ eventually and therefore $\sqrt{x^4}=x^2$. Then, we can write

$$\frac{x^2}{\sqrt{x^4+1}}=\frac{1}{\sqrt{1+1/x^4}}.$$

Using the fact that $\lim_{x\to\infty}\frac{1}{x^r}=0$, we see that

$$\lim_{x\to\infty}\left(1+\frac{1}{x^4}\right)=1$$

and by the root law for limits, we conclude that

$$\lim_{x\to\infty}\sqrt{1+\frac{1}{x^4}}=1.$$

By the quotient law, we obtain

$$\lim_{x\to\infty}\frac{x^2}{\sqrt{x^4+1}}=\frac{1}{1}=1.$$

Problem 18

We have

$$\sqrt{1+4x^6}=\sqrt{x^6(1/x^6+4)}=|x|^3\sqrt{1/x^6+4}.$$

Now, since $x < 0$, we have $|x| = -x$ and so

$$\sqrt{1+4x^6} = -x^3\sqrt{1/x^6+4}.$$

Then, we can rewrite the limit and compute it:

$$\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to-\infty}\frac{-x^3\sqrt{1/x^6+4}}{x^3(2/x^3-1)} = \lim_{x\to-\infty} -\frac{\sqrt{1/x^6+4}}{2/x^3-1} = -\frac{\sqrt{\lim_{x\to-\infty} 1/x^6+4}}{\lim_{x\to-\infty} 2/x^3-1} = 2.$$

Problem 21

We multiply by the conjugate:

$$(\sqrt{9x^2+x}-3x)\left(\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}\right) = \frac{x}{\sqrt{9x^2+x}+3x}$$

and then factor x:

$$\frac{x}{\sqrt{9x^2+x}+3x} = \frac{x}{\sqrt{x^2}\sqrt{9+1/x}+3x}.$$

Since we are taking the limit as $x\to\infty$, we have $\sqrt{x^2} = x$. Therefore,

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) = \lim_{x\to\infty}\frac{x}{x\sqrt{9+1/x}+3x} = \lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}.$$

Since $\lim_{x\to\infty}\frac{1}{x} = 0$, then by the rule for limits, we get

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

Problem 22

Since $x\to-\infty$, we have that $x < 0$ eventually. Let's multiply by the conjugate:

$$\left(\sqrt{4x^2+3x}+2x\right)\left(\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2-3x}-2x}\right) = \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}-2x} = \frac{3x}{\sqrt{4x^2+3x}-2x}.$$

Factoring x^2 in the root, we find

$$\sqrt{4x^2+3x} = \sqrt{x^2}\sqrt{4+3/x}$$

and since $x < 0$, we have $\sqrt{x^2} = -x$. This means we can rewrite the above expression as followed:

$$\sqrt{4x^2+3x} = -x\sqrt{4+3/x}.$$

Replacing this last expression in the quotient above, we find out that

$$\sqrt{4x^2+3x}+2x = \frac{3}{-\sqrt{4+3/x}-2}.$$

We therefore see that

$$\lim_{x\to-\infty}\left(-\sqrt{4+\frac{3}{x}}-2\right) = -4.$$

Therefore, we obtain

$$\lim_{x\to-\infty}\frac{3}{\sqrt{4+3/x}-2} = \frac{3}{-4} = -\frac{3}{4}.$$

Problem 30

We factor x^2:

$$x^2 - x^4 = x^2(1-x^2).$$

Therefore, since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty}(1-x^2) = -\infty$, we obtain

$$\lim_{x\to\infty}(x^2-x^4) = \lim_{x\to\infty} x^2 \lim_{x\to\infty}(1-x^2) = \infty(-\infty) = -\infty.$$

Problem 40

We first compute the limit at ∞. Since $x \to \infty$, the variable x will be eventually positive and so $\sqrt{x^2} = x$. Then, we can write

$$\frac{x-9}{\sqrt{4x^2+3x+2}} = \frac{1-9/x}{\sqrt{4+3/x+2/x^2}}.$$

We then find, by the quotient rule and the root rule,

$$\lim_{x\to\infty}\frac{1-9/x}{\sqrt{4+3/x+2/x^2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

So $y = 1/2$ is an HA at ∞.

Finally, we compute the limit at $-\infty$. Since $x \to -\infty$, the variable x will eventually be negative and so $\sqrt{x^2} = -x$. We then can write

$$\frac{x-9}{\sqrt{4x^2+3x+2}} = -\frac{1-9/x}{\sqrt{4+3/x+2/x^2}}.$$

We then find, by the quotient rule and the root rule,

$$\lim_{x\to\infty} -\frac{1-9/x}{\sqrt{4+3/x+2/x^2}} = \frac{1}{\sqrt{4}} = -\frac{1}{2}.$$

So $y=-1/2$ is an HA at $-\infty$.

Problem 46

We need the function to have a non-zero numerator and a zero denominator at $x=1$ and $x=3$. A good function for that would be

$$\frac{1}{(x-1)(x-3)}.$$

This last expression, however, does not have a horizontal asymptote $y=1$ because

$$\lim_{x\to\infty} \frac{1}{(x-1)(x-3)} = 0$$

and

$$\lim_{x\to-\infty} \frac{1}{(x-1)(x-3)} = 0.$$

To change this, we can add 1 to the previous expression. The desire function is therefore

$$f(x) = 1 + \frac{1}{(x-1)(x-3)}.$$

Problem 63

We will use the Squeeze Theorem. First, we have

$$\lim_{x\to\infty} \frac{4x-1}{x} = 4.$$

Secondly, we have

$$\lim_{x\to\infty} \frac{4x^2+3x}{x^2} = 4.$$

Therefore, the function f is squeeze between two functions that have 4 as their limit at ∞. Using the Squeeze Theorem, we conclude that

$$\lim_{x\to\infty} f(x) = 4.$$