Question 1

A bottle rocket was equipped with an altimeter and launched directly upward. Its altitude was recorded every second in the following table.

Time (seconds)	Altitude (meters)
0	0
1	4
2	9
3	15
4	22
5	30
6	38

* Comment:
−1/4 per arithmetic error. No further points taken off if work is consistent with error.

(a) (1 points) What was the bottle rocket's average velocity between 2 and 5 seconds?

(1 pt) $V_{[2,5]} = \frac{30-9}{3} = \frac{21}{3} = 7$ m/s

(b) (2 points) Is the rocket moving faster, on average, between 2 and 5 seconds or between 5 and 6 seconds?

(1pt) $V_{[5,6]} = \frac{38-30}{1} = 8$ m/s

(1pt) $V_{[5,6]} > V_{[2,5]}$, so rocket moved faster on average between 5 and 6 seconds.

(c) (3 points) Estimate the bottle rocket's instantaneous velocity at 4 seconds. Make sure the method you are using is clear, and include units in your answer.

Estimate instantaneous velocity at t=4 by averaging the average velocity over [3,4] and [4,5]

(1pt) $V_{[3,4]} = \frac{22-15}{1} = 7$ m/s

(1pt) $V_{[4,5]} = \frac{30-22}{1} = 8$ m/s

instantaneous velocity at t=4 is approximately

$\frac{V_{[3,4]} + V_{[4,5]}}{2} = \frac{7+8}{2} = \frac{15}{2}$ m/s

(1pt)

$\boxed{\text{or any logical argument}}$

Midterm 1

Name (Last, First): ________________________ Section: __________

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	16	14	16	5	17	16	10	100
Score:									

Instructions:

- Write your full name (last, first) and section number above.
- Answer all the questions below and show your work.
- No electronic devices are to be used during the exam (this includes calculators).
- The exam is closed book and closed notes.
- **Do not use L'Hôpital's rule anywhere on this exam.**
- Turn in your exam at the end of the period.

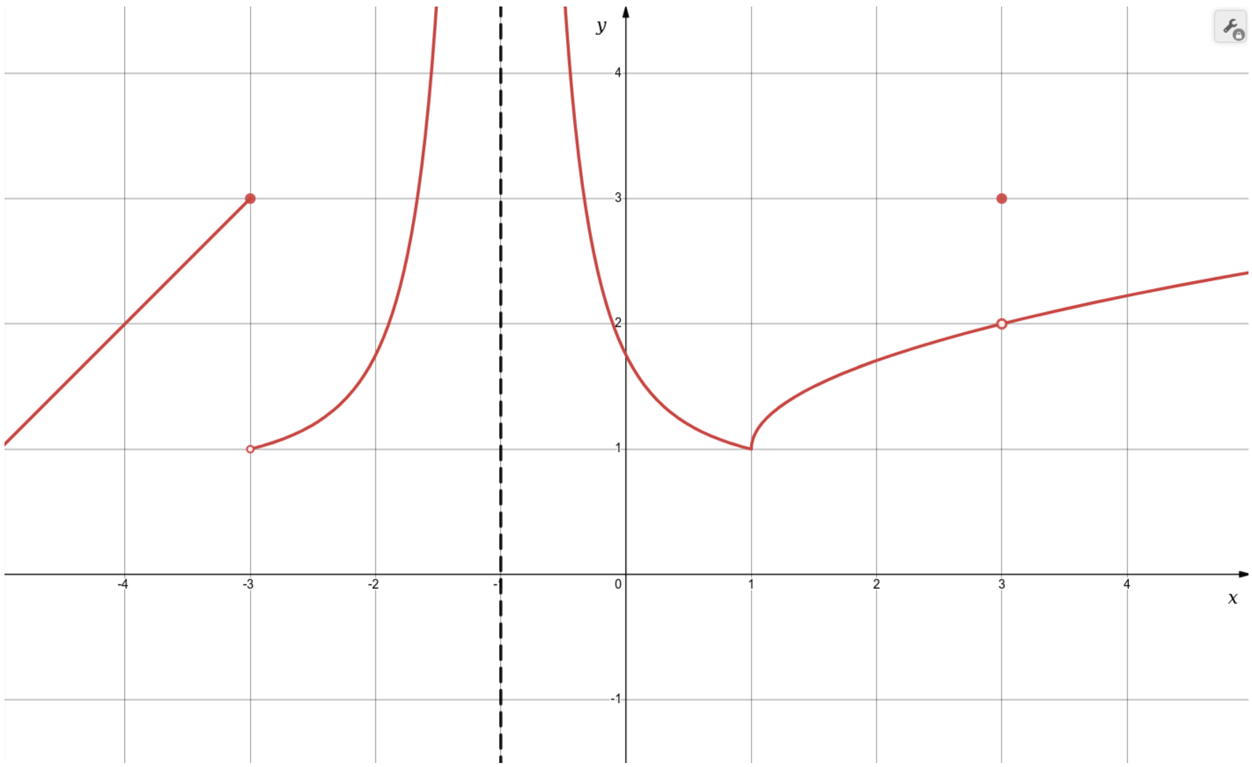
Good luck!

Sign below to acknowledge that you have read and agree to the above instructions.

Signature: ________________________________

Question 2

The graph of a function f is shown below. Assume f has a vertical asymptote at $x = -1$.



(a) (6 points) Evaluate each of the following limits. If the limit is ∞ or $-\infty$, specify which. If the limit does not exist and is not ∞ or $-\infty$, then write DNE.

each 1 pt

1. $\lim_{x\to 3} f(x)$ = 2
2. $\lim_{x\to -3} f(x)$ = DNE
3. $\lim_{x\to -1^+} f(x)$ = ∞
4. $\lim_{x\to -3^-} f(x)$ = 3
5. $\lim_{x\to -3^+} f(x)$ = 1
6. $\lim_{x\to -1} f(x)$ = ∞

(b) (3 points) For which values (if any) in the interval $[-4, 4]$ is the function f not continuous?

$x = -3, -1, 3$ 1pt for each correct answer

(c) (3 points) For which values (if any) in the interval $[-4, 4]$ is the function f continuous, but not differentiable?

$x = 1$ 1 point for this answer
+ 2pts for no wrong answers

(d) (4 points) On which intervals is the function f decreasing? On which intervals is f increasing?

decreasing: $(-1, 1)$ + 2pts

increasing: $[-4, -3), (-3, -1), (1, 3), (3, 4]$
or
$(-\infty, -3), (-3, -1), (1, 3), (3, \infty)$ + 2pts

Question 3

Evaluate the following limits. If the limit does not exist, write DNE.

(a) (2 points) $\lim_{x\to 1} (2x^3 + x^2 - 3)$

(1 pt) $= 2\cdot\lim_{x\to 1} x^3 + \lim_{x\to 1} x^2 - 3\cdot\lim_{x\to 1} 1$

(1 pt) $= 2\cdot 1^3 + 1^2 - 3\cdot 1 = \boxed{0}$

OR

$\lim_{x\to 1} (2x^3+x^2-3)$

(1 pt) polynomial ⇒ can use Direct Subst.

$= 2\cdot 1^3 + 1^2 - 3 = \boxed{0}$ (1 pt)

(b) (4 points) $\lim_{x\to 1} \sqrt{x^2+8}$.

(2 pts) $= \sqrt{\lim_{x\to 1} x^2+8}$

(1 pt) $= \sqrt{\lim_{x\to 1} x^2 + \lim_{x\to 1} 8}$

(1 pt) $= \sqrt{1^2+8} = \sqrt{9} = \boxed{3}$

OR

$= \sqrt{\lim_{x\to 1} x^2+8}$ (2 pts)

$= \sqrt{1^2+8} = \sqrt{9} = \boxed{3}$ (1 pt)

polynomial ⇒ can use Direct Subst. (1 pt)

(c) (4 points) $\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$. $= \frac{2^2-3.2+2}{2-2} = \frac{0}{0}$, by Direct Substitution (1 pt)

(2 pts) $= \lim_{x\to 2} \frac{(x-1)\cancel{(x-2)}}{\cancel{x-2}}$

(1 pt) $= \lim_{x\to 2} x-1 = 2-1 = \boxed{1}$

(d) (4 points) $\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$. $= \frac{0}{\sqrt{0+1}-1} = \frac{0}{0}$ by Direct Substitution (1 pt)

(1 pt) $= \lim_{x\to 0} \frac{x}{(\sqrt{x+1}-1)} \times \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$

(1 pt) $= \lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{x+1-1} = \lim_{x\to 0} \frac{\cancel{x}(\sqrt{x+1}+1)}{\cancel{x}}$

(1 pt) $= \lim_{x\to 0} \sqrt{x+1}+1 = \sqrt{1}+1 = \boxed{2}$

Question 4

Let $f(x) = \dfrac{1}{x^2}$.

(a) (12 points) Use the definition of the derivative (that is the limit process) to compute $f'(x)$. **You will not get any credit unless you use the definition of the derivative.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ 1pt. Definition

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$ 1pt. Expr. of f.

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{(x+h)^2 x^2 h}$$ 4pts. Same Denom.

$$= \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{(x+h)^2 x^2 h}$$ 2pts. Expand Square.

$$= \lim_{h \to 0} \frac{-2x - h}{(x+h)^2 x^2}$$ 2pts. Simplifying h.

$$= \frac{-2x}{x^4}$$ 2pts. Eval. limit.

So, $f'(x) = \dfrac{-2}{x^3}$

(b) (4 points) Use part (a) to find the equation of the tangent line to the curve $y = f(x)$ at the point $(1, 1)$.

Point slope equation: $y - 1 = f'(1)(x-1)$ 1pt. Eq. tg. line.

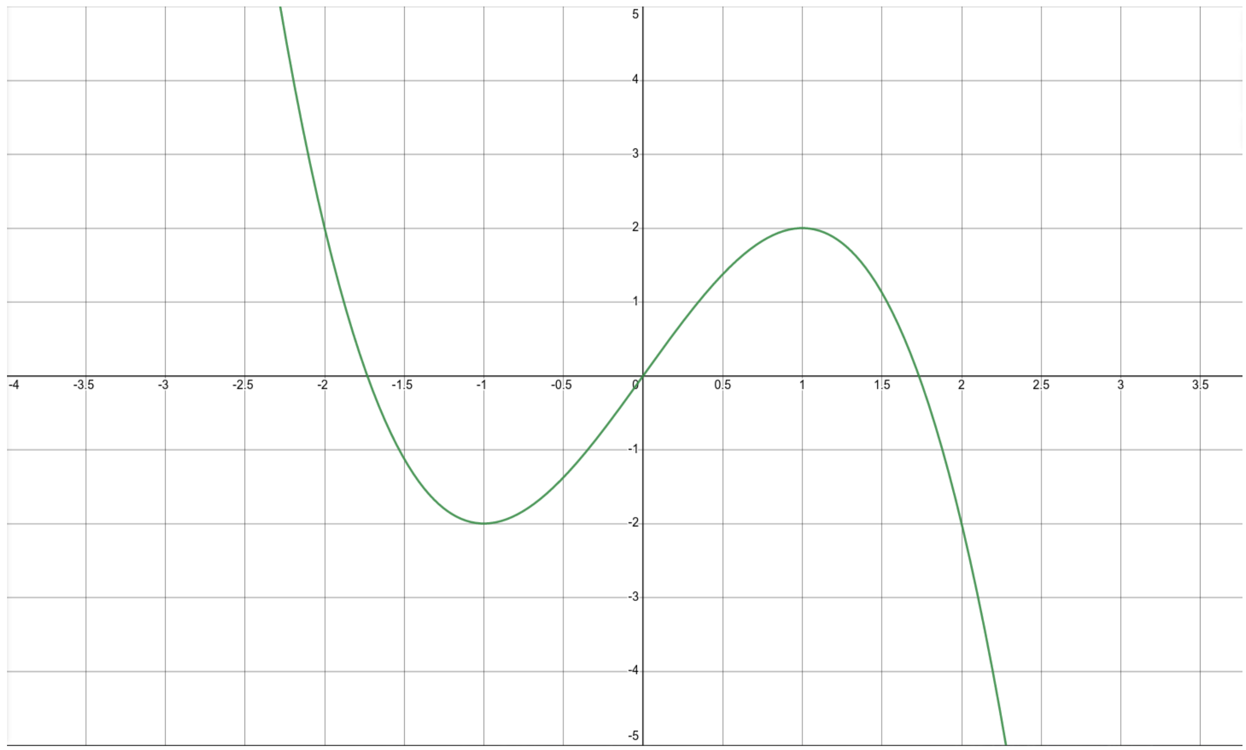
From (a), $f'(1) = \dfrac{-2}{1} = -2$. 2pts. Eval. derivative

So, $\boxed{y = -2x + 3}$ 1pt. Give the answer

Remark: can leave as the point-slope form.

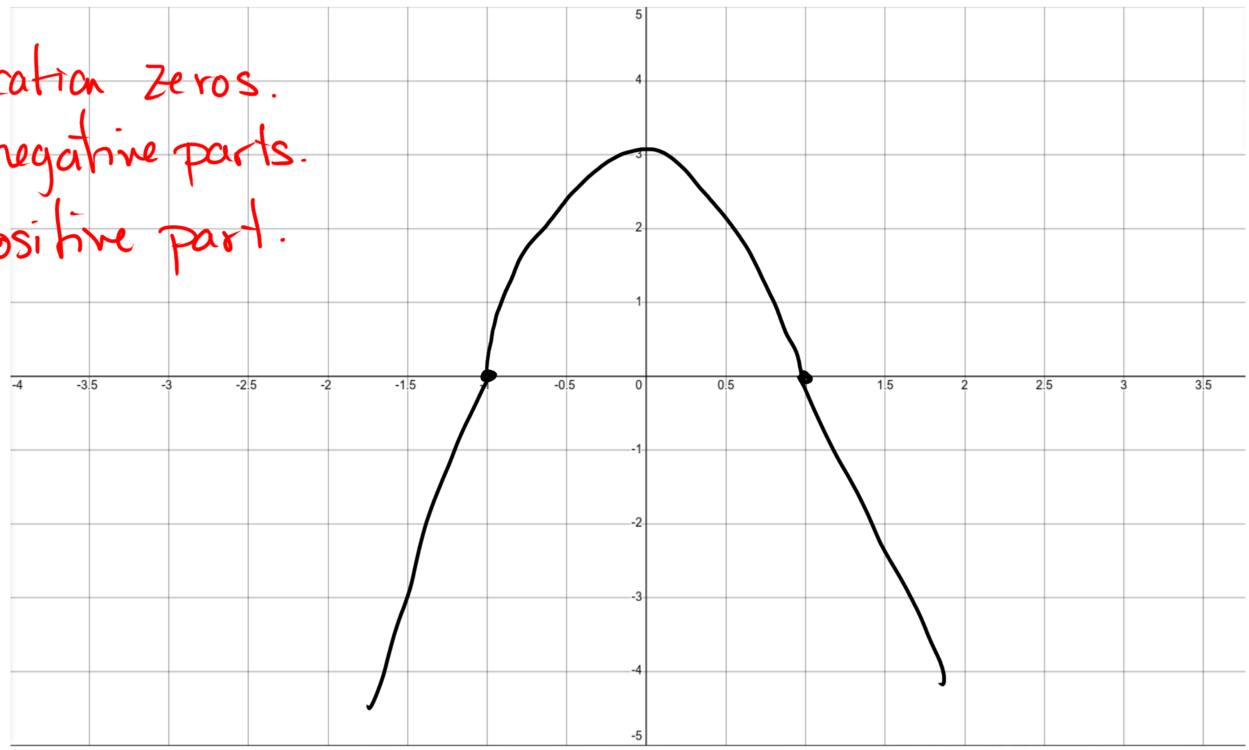
Question 5

The graph $y = f(x)$ is shown below.



(5 points) Sketch the graph of $f'(x)$ below, using the graph of $y = f(x)$.

2pts: location zeros.
2pts: negative parts.
1pt: positive part.



Question 6

For each of the functions f below, compute $f'(x)$. Do not simplify your answer after applying the differentiation rules.

(a) (2 points) $f(x) = 3x^2 + 2x - 6$

$$f'(x) = 6x + 2$$

+ 2pts

(b) (5 points) $f(x) = \frac{2x^3 + 4x}{x - 2}$

$$f'(x) = \frac{\overbrace{(6x^2+4)}^{1\text{pt}}(x-2) - \overbrace{(1)}^{1\text{pt}}(2x^3+4x)}{(x-2)^2}$$

+3 pts for quotient rule

(c) (5 points) $f(x) = 5\sqrt{x}(x^2 + 3x - 1)$

$$f'(x) = 5\left(\frac{1}{2}x^{-1/2}(x^2+3x-1) + \sqrt{x}(2x+3)\right)$$

+3pts for product rule

or

$$f'(x) = \underbrace{\left(5 \cdot \frac{1}{2}x^{-1/2}\right)}_{+1\text{pt}}(x^2+3x-1) + 5\sqrt{x}\underbrace{(2x+3)}_{+1\text{pt}}$$

(d) (5 points) $f(x) = \sqrt[3]{x^2 + x + 1}$

$$f'(x) = \frac{1}{3}\underbrace{(x^2+x+1)}_{\substack{\text{unchanged}\\ +1\text{pt}}}{}^{-2/3} \cdot \underbrace{(2x+1)}_{+1\text{pt}}$$

+ 3pts for chain rule

Question 7

For each of the functions below, compute its derivative. Do not simplify your answer after applying the differentiation rules.

(a) (8 points) $f(t) = \dfrac{\sin(4t) + \tan(t)}{t^3 - 2}$

$$f'(t) = \frac{(t^3-2)(\sin 4t + \tan t)' - (\sin 4t + \tan t)(t^3-2)'}{(t^3-2)^2}$$

3 pts

$$= \frac{(t^3-2)\left((\sin 4t)' + (\tan t)'\right) - (\sin 4t + \tan t)(t^3-2)'}{(t^3-2)^2}$$

2 pts, 2 pts, 1 pt

$$= \frac{(t^3-2)(4\cos 4t + \sec^2 t) - (\sin 4t + \tan t)(3t^2)}{(t^3-2)^2}$$

(b) (8 points) $g(x) = \sin\left((4x-3)^2\right)$

$$g'(x) = \cos\left((4x-3)^2\right) \frac{d}{dx}\left((4x-3)^2\right)$$

$$= \cos\left((4x-3)^2\right) \cdot 2(4x-3) \cdot \frac{d}{dx}(4x-3)$$

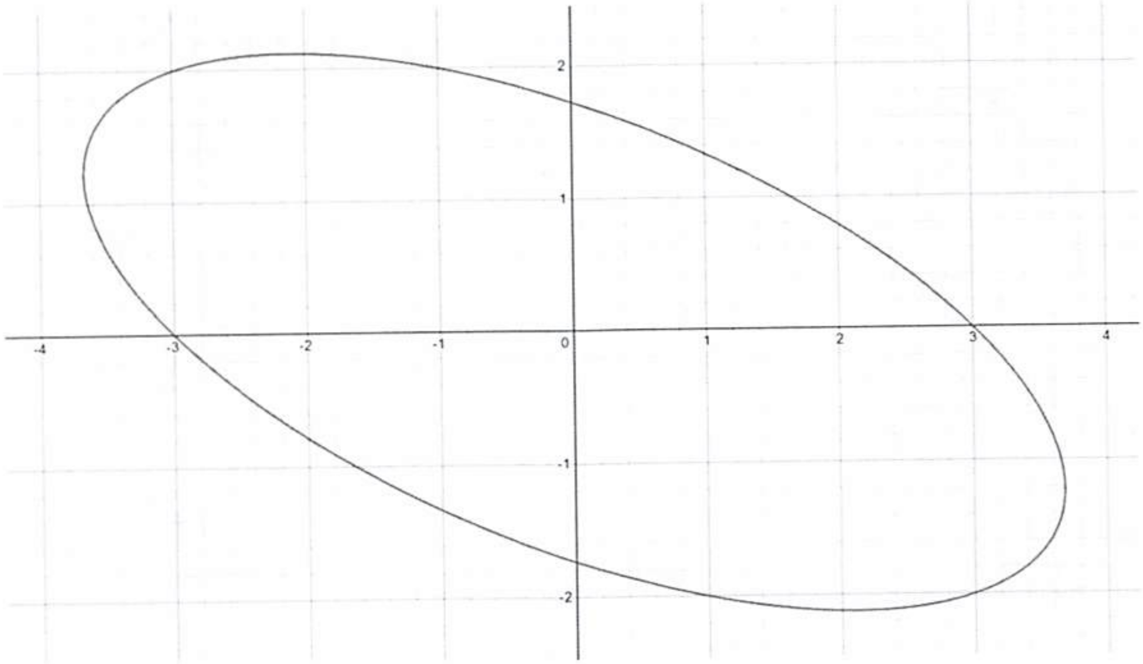
$$= \cos\left((4x-3)^2\right) \cdot 2(4x-3) \cdot 4$$

2 pts, 2 pts, 2 pts

(I think there should be 2 pts for any reasonable effort to get the total to 8 pts.

Question 8

Consider the equation $x^2 + 2xy + 3y^2 = 9$, whose graph is the ellipse shown below.



(10 points) Find the equation of the tangent line to the curve at the point $(3, -2)$. Put your answer in the form $y = mx + b$.

(1 pt) $\therefore \frac{d}{dx}\left[x^2+2xy+3y^2\right] = \frac{d}{dx}[9]$

(1 pt) $\therefore \frac{d}{dx}[x^2] + 2\frac{d}{dx}[xy] + 3\frac{d}{dx}[y^2] = 0$

(4 pts) $\therefore 2x + 2\left(x\frac{d}{dx}[y] + y\frac{d}{dx}[x]\right) + 3\frac{d}{dy}[y^2]\cdot\frac{dy}{dx} = 0$

(1 pt) $\therefore 2x + 2\left(x\frac{dy}{dx} + y\right) + 6y\frac{dy}{dx} = 0$

When $x=3$, $y=-2$: $6 + 2\left(3\frac{dy}{dx} - 2\right) - 12\frac{dy}{dx} = 0$ (1 pt)

$\therefore \boxed{\frac{dy}{dx} = \frac{1}{3}}$

Equation of Tangent line: $y - y_0 = m(x - x_0)$, $m = \frac{dy}{dx}$

(1 pt) $\therefore y + 2 = \frac{1}{3}(x-3)$

(1 pt) $\therefore \boxed{y = \frac{1}{3}x - 3}$