MATH-241 Calculus I
Midterm 01

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Spring 02/17/2022

Last name: Solution
First name: —

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	15	15	20	10	10	15	9	100
Score:	—	—	—	—	—	—	—	—	—

Instructions: Make sure to write your complete name on your copy. You must answer all the questions below and write your answers directly on the questionnaire. At the end of the 75 minutes, hand out your copy.

No devices such as a smart phone, cell phone, laptop, or tablet can be used during the exam. You are not allowed to use the lecture notes, the textbook, or any other notes. You may use a digital calculator (no graphical calculator or symbolic calculator will be allowed).

You must show ALL your work to have full credit. An answer without justification worth no point.

Good luck!

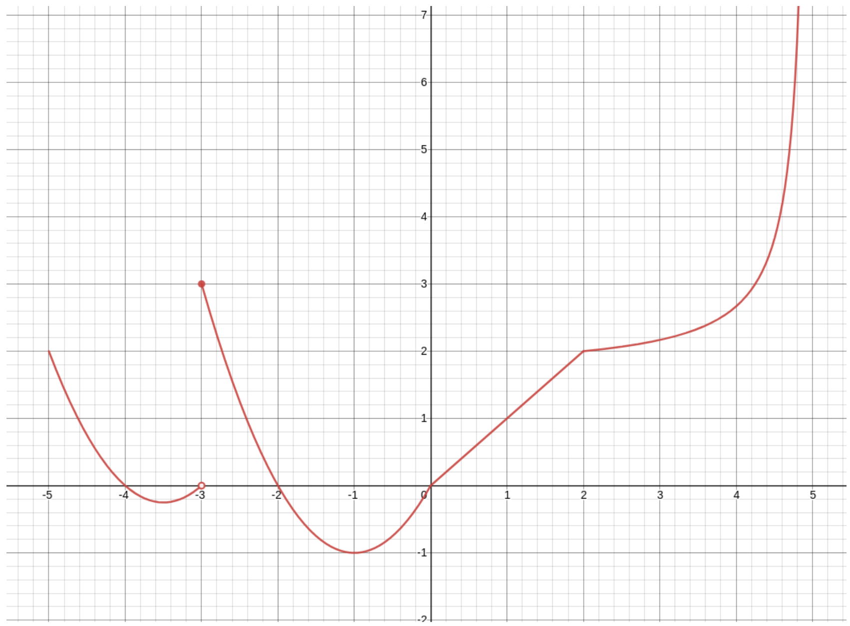
Pierre-Olivier Parisé

Your Signature:

University of Hawai'i

QUESTION 1 **(6 pts)**

Consider the function $f(x)$ with the graph $y = f(x)$ pictured below. The domain of f is $[-5, 5]$.



(a) (1 point) On which interval(s) (if any) is the function decreasing? (no justification needed)

$[-5, -3.5] \cup [-3, -1]$.

(b) (1 point) Where (if anywhere) is the function not continuous?

At $x = -3$ & $x = 5$

(c) (1 point) Where (if anywhere) is the function not differentiable?

at $x = -5$, $x = -3$; $x = 0$, $x = 2$ & $x = 5$.

(d) (1 point) What is $\lim_{x \to -3^-} f(x)$?

it is 0 .

(e) (1 point) What is $\lim_{x \to 5^-} f(x)$?

It is $+\infty$.

(f) (1 point) What is $\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$? $\rightarrow \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = 1$.

$\frac{f(x) - f(2)}{x - 2}$ is the slope of the line segment which is $\underline{1}$ for any x .

QUESTION 2 **(15 pts)**

Find the value of the following limit. No credit will be attributed for using L'Hôpital's rule to find the value of a limit.

(a) (5 points) $\lim_{x\to 0} \frac{x^2-4}{x-2}$.

$$\lim_{x\to 0} \frac{x^2-4}{x-2} = \lim_{x\to 0} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x\to 0} x+2$$

$$= \boxed{2}$$

(b) (5 points) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\cancel{(\sqrt{x}-3)}}{\cancel{(\sqrt{x}-3)}(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \boxed{1/6}$$

(c) (5 points) $\lim_{x\to 0} \left(\frac{\cos x - 1}{x}\right)^2 = L$

$\frac{1-\cos x}{2} = \sin^2\left(\frac{x}{2}\right)$ so that

$$L = \lim_{x\to 0} \left(\frac{2\sin^2\left(\frac{x}{2}\right)}{x}\right)^2 = \left[\lim_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{x/2} \sin\left(\frac{x}{2}\right)\right]^2$$

$$= \left[\lim_{x\to 0} \frac{\sin(x/2)}{x/2} \lim_{x\to 0} \sin\left(\frac{x}{2}\right)\right]^2 = (1\cdot 0)^2 = \boxed{0}$$

Question 3 (15 pts)

Let $f(x) = 1/x$.

(a) (5 points) State the definition of the derivative of a function at some point a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(b) (10 points) Using the definition of the derivative, find the value of $f'(3)$ if $f(x) = 1/x$. **No credit for a solution using the rules of differentiation.**

$$f'(3) = \lim_{h \to 0} \frac{1/(3+h) - \frac{1}{3}}{h}$$

$$= \lim_{h \to 0} \frac{3 - (3+h)}{(3+h)h\,3}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{(3+h)\cancel{h}\,3}$$

$$= \lim_{h \to 0} \frac{-1}{(3+h)3}$$

$$= \boxed{\frac{-1}{9}}$$

Question 4 **(20 pts)**

Find the equation of the tangent line to the curve $y = \frac{x^2+5}{x-2}$ at the point $(1, -6)$.

Tangent line at $(1,-6)$:

$$y+6 = f'(1)(x-1).$$

We have

$$f'(x) = \frac{(x^2+5)'(x-2) - (x^2+5)(x-2)'}{(x-2)^2}$$

$$= \frac{2x(x-2) - (x^2+5)}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2 - 5}{(x-2)^2}$$

$$= \frac{x^2 - 4x - 5}{(x-2)^2}$$

$$= \frac{(x-5)(x+1)}{(x-2)^2}$$

$$\Rightarrow \quad f'(1) = \frac{(-4)(2)}{(-1)^2} = -8.$$

So,

$$y+6 = -8(x-1)$$

$$\Rightarrow \boxed{y = -8x+2}$$

Question 5 **(10 pts)**

An 'ilio-holo-ika-uaua[1] is fitted with a GPS to track his movement. She gets in the ocean, and swims in a straight line away from the shore. Her position from the shore is recorded every two minutes for ten minutes. The results are recorded in the following table.

Time in minutes	Distance from the shore in meters
0	0
2	20
4	30
6	60
8	80
10	100

(a) (3 points) What is the seal's average velocity between minutes 4 and 8?

$$\frac{\Delta x}{\Delta t} = \frac{80-30}{8-4} = \frac{50}{4} = \boxed{\frac{25}{2} \text{ m/s}}$$

$\hookrightarrow$ 12.5 m/s

(b) (4 points) Estimate the seal's velocity at time 6 minutes.

average 1: $\frac{60-30}{6-4} = 15 \text{ m/s} \approx v(6)$

average2: $\frac{80-60}{8-6} = 10 \text{ m/s} \approx v(6)$

Or, better estimate: $v(6) \approx \frac{15+10}{2} = \boxed{12.5 \text{ m/s}}$

(c) (3 points) The seal saw a group of fish at time 6 minutes going at 10 m/min. Do you think that the seal could catch the fish?

According to (b), the seal could catch the school of fishes.

[1] This means "dog running in the rough water" in Hawaiian. The name of the animal in English is a hawaiian monk seal.

Question 6 (10 pts)

Consider the function $f(x) = 4x^3 - 6x^2 - 6x + 5$. This function must have at least one zero in the interval $(0, 1)$. Explain why, making explicit which theorem(s), if any, and which assumtions(s) on f, if any, you are using.

f polynomial $\rightarrow$ continuous on $(-\infty, \infty)$

$\rightarrow$ continuous on $[0, 1]$.

We have

- $f(0) = 5$
- $f(1) = 4 - 6 - 6 + 5 = -1$
- $-1 < \underset{N}{\underset{\shortparallel}{0}} < 5$

So, from the Intermediate Value Theorem, there is a c between 0 & 1 such that

$f(c) = 0$.

QUESTION 7 **(15 pts)**

Compute the derivatives of the following functions.

(a) (5 points) $f(x) = \dfrac{\sin x}{x^3 + \cos x}$.

$$f'(x) = \frac{\cos x\,(x^3+\cos x) - \sin x\,(3x^2 - \sin x)}{(x^3+\cos x)^2}$$

$$= \frac{x^3\cos x + \cos^2 x - 3x^2\sin x + \sin^2 x}{(x^3+\cos x)^2}$$

$$= \boxed{\frac{x^3\cos x - 3x^2\sin x + 1}{(x^3+\cos x)^2}.}$$

(b) (5 points) $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$.

$$f'(x) = \frac{-\frac{1}{2\sqrt{x}}(1+\sqrt{x}) - (1-\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2}$$

$$= \frac{-1-\sqrt{x} - 1+\sqrt{x}}{2\sqrt{x}\,(1+\sqrt{x})^2}$$

$$= \boxed{\frac{-1}{\sqrt{x}\,(1+\sqrt{x})^2}}$$

(c) (5 points) $f(x) = x\sqrt{x}$.

$$f'(x) = \frac{d}{dx}\left(x\sqrt{x}\right)$$

$$= \frac{d}{dx}\left(x^{3/2}\right)$$

$$= \frac{3}{2}x^{1/2} = \boxed{\frac{3\sqrt{x}}{2}}$$

Question 8 **(9 pts)**

Answer each of the following questions. No credit will be attributed for using L'Hôpital's rule to find the value of a limit.

(a) (3 points) Let $f(x) = \begin{cases} Ax & x \leq -1 \\ x^2 - 3Ax + 3 & x > -1. \end{cases}$

Find the value of A for which the function f is continuous.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \quad \left(\text{limit should exist} = f(-1)\right)$$

$$\Rightarrow \lim_{x \to -1^-} Ax = \lim_{x \to -1^+} x^2 - 3Ax + 3$$

$$\Rightarrow \quad -A = (-1)^2 + 3A + 3$$

$$\Rightarrow \quad -4A = 4 \quad \Rightarrow \boxed{A = -1}.$$

(b) (3 points) If $g(x) = (x+1)f(x)$ and $\lim_{x \to 0} f(x) = 2$, then find $\lim_{x \to 0} g(x)$.

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} (x+1) \lim_{x \to 0} f(x)$$

$$= 1 \cdot 2$$

$$= \boxed{2}.$$

(c) (3 points) Find the value of $\lim_{x \to 3^+} \dfrac{x+4}{x-3}$.

$$\lim_{x \to 3^+} \frac{x+4}{x-3} = \frac{3^+ + 4}{3^+ - 3} = \frac{7^+}{0^+} = \boxed{+\infty}$$