MATH-241
Midterm 02

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Last name: Solutions

First name: [redacted]

Section: [redacted]

Instructions:

- Make sure to write your complete name on your copy.
- You must answer all eight (8) questions below and write your answers directly on the questionnaire.
- You have 75 minutes to complete the exam.
- When you are done (or at the end of the 75min period), return your copy.
- Devices such as smartphones, cellphones, laptops, tablets, e-readers, ipods, gameboys (and, you know, any other electronic devices that I haven't thought of) may not be used during the exam.
- You can not use a calculator.
- **Turn off your cellphones during the exam**.
- Lecture notes and the textbook are not allowed during the exam.
- You must show ALL your work to have full credit. An answer without justification is worth no points (except if it is mentioned explicitly in the question not to justify).
- Draw a square around your final answer.

Your Signature: [redacted]

May the Force be with you!

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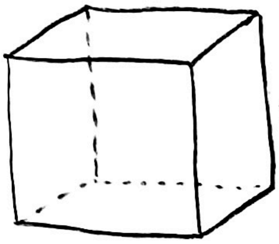
Question 1 (10 pts)

The volume of a cube is increasing at a rate of $10\text{cm}^3/\text{min}$. How fast is the surface area increasing when the length of an edge is 30cm?



x: side length (cm)

1pt. A: surface area (cm^2)

V: Volume (cm^3).

Given: $V = x^3$ $\quad \frac{dV}{dt} = 10$

3pts. $A = 6x^2$ $\quad$ 5pts.

Goal: 1pt. Find $\left.\frac{dA}{dt}\right|_{x=30}$.

① Find $\frac{dx}{dt}$

5pts.

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow 10 = 3 \cdot 30^2 \frac{dx}{dt}$$

2 pts So, $\frac{dx}{dt} = \frac{1}{270}$ cm/min.

② Find $\frac{dA}{dt}$

$$\frac{dA}{dt} = 12x \frac{dx}{dt}$$

3pts. $\Rightarrow \left.\frac{dA}{dt}\right|_{x=30} = 12 \cdot 30 \cdot \frac{1}{270} = \frac{4 \cdot 9 \cdot 10}{270}$

$$\Rightarrow \boxed{\left.\frac{dA}{dt}\right|_{x=30} = \frac{4}{3} \text{ cm}^2/\text{min}}.$$

QUESTION 2 (10 pts)

Let $f(x) = \sqrt[3]{1+3x}$.

(a) (5 points) Find the linearization of $f(x)$ at $a = 0$.

$f'(x) = (1+3x)^{-2/3} \Rightarrow f'(0) = 1$

$f(0) = 1$.

So, $L(x) = f'(0)(x-0) + f(0)$

$\Rightarrow L(x) = x + 1$.

(b) (5 points) Use the linearization to approximate the value of $\sqrt[3]{1.03}$.

$1.03 = 1 + 3x \Leftrightarrow x = 0.01$.

So, $\sqrt[3]{1.03} \approx L(0.01) = 0.01 + 1$

$\Rightarrow \sqrt[3]{1.03} \approx 1.01$.

QUESTION 3 (20 pts)

Let $f(x) = \dfrac{x}{1-x^2}$.

(a) (4 points) Using **Calculus**, find the vertical asymptotes (if any) and horizontal asymptotes (if any) of the function $f(x)$.

VA: $1-x^2 = (1-x)(1+x) \to \boxed{x=-1}$ & $\boxed{x=1}$

$$\lim_{x\to 1^-} f(x) = \frac{1}{(1+1)(1-1^-)} = \frac{1}{2\cdot 0^+} = +\infty$$

$$\lim_{x\to 1^+} f(x) = \frac{1}{(1+1)(1-1^+)} = \frac{1}{2\cdot 0^-} = -\infty$$

3pts.

$$\lim_{x\to -1^-} f(x) = \frac{-1}{(1+(-1)^-)(1-(-1)^-)} = \frac{-1}{0^-\cdot 2} = +\infty$$

$$\lim_{x\to -1^+} f(x) = \frac{-1}{(1+(-1)^+)(1-(-1)^+)} = \frac{-1}{0^+\cdot 2} = \infty$$

HA: $\lim_{x\to\pm\infty} \dfrac{x}{1-x^2} = 0$, $\boxed{y=0}$ 1pts.

(b) (4 points) The first derivative of f is $f'(x) = \frac{1+x^2}{(x^2-1)^2}$. Find the critical numbers (if any) and the interval(s) of increase and decrease.

Dom $f' = (-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

CN are $x=\pm 1$ because $f'(x)\neq 0$ for all x. 1pts

factors		-1		1	
$(x-1)^2$	+		+		+
$(x+1)^2$	+		+		+
$f'(x)$	+	$\nexists$	+		+
$f(x)$	↗ 1pt	VA	↗ 1pt	VA	↗ 1pt.

(c) (4 points) The second derivative of f is $f''(x) = -\frac{2x(3+x^2)}{(x^2-1)^3}$. Find the x-coordinate of the inflection points (if any) and the interval(s) of concavity.

I.P. $f''(x) = 0 \iff x = 0$. ✓ 1pt.

$f''(x)$ DNE if $x = \pm 1 \to$ not in the domain, not Inflection point.

factors		-1		0		1	
$-2x$	$+$		$+$		$-$		$-$
$(x-1)^3$	$-$		$-$		$-$		$+$
$(x+1)^3$	$-$		$+$		$+$		$+$
$f''(x)$	$+$	$\nexists$	$-$	0	$+$	$\nexists$	$-$
$f(x)$	$\smile$		$\frown$		$\smile$		$\frown$

3pts.

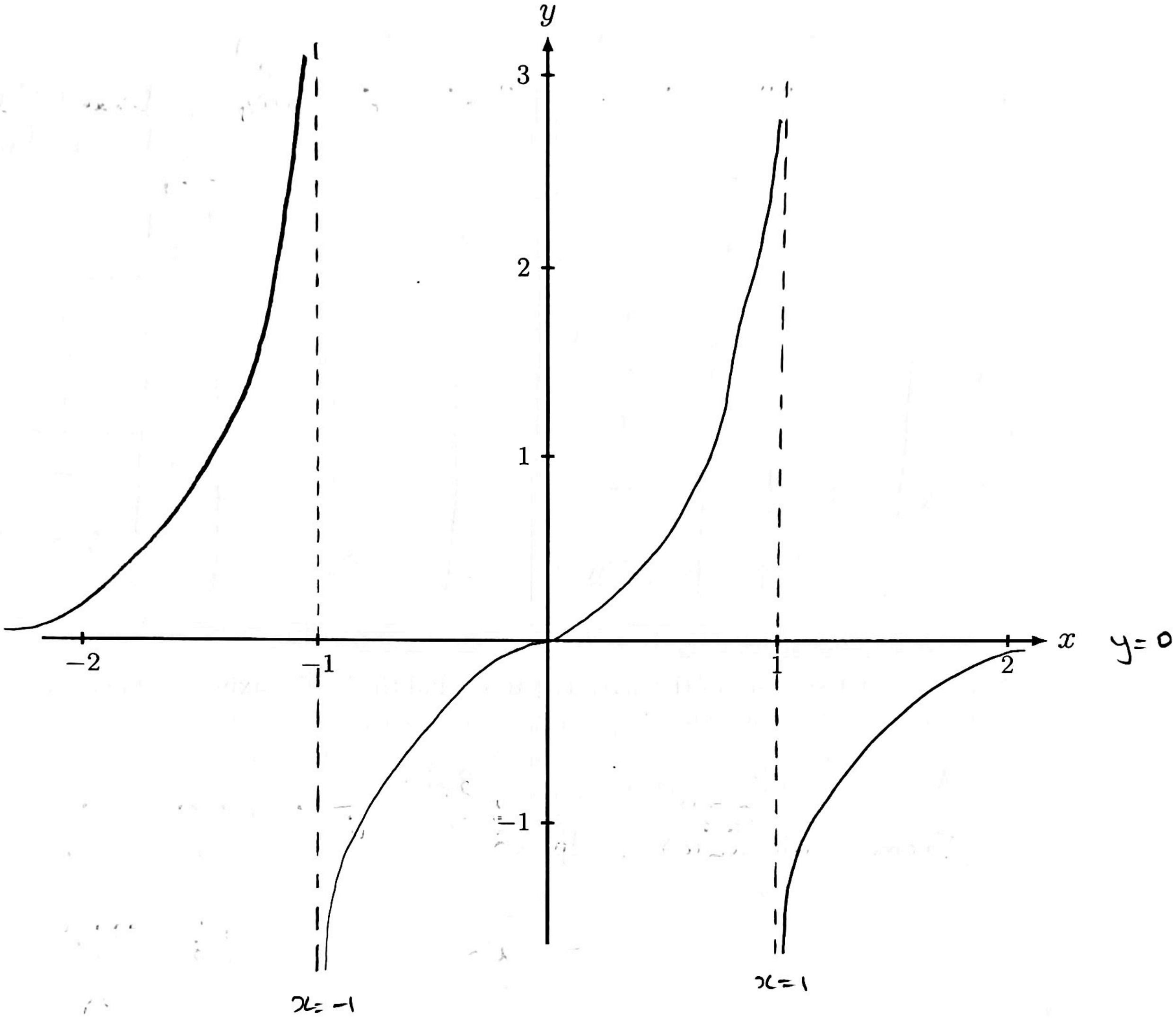
(d) (4 points) Using one of the derivative tests, find the local maximum(s) and/or local minimum(s) of the function.

No local maximum 3pt.

From (b) & (c). 1pt.

...Question 3 continued...

(e) (4 points) Sketch the graph of the function f in the axes below.



QUESTION 4 (10 pts)

Compute the following limits. If the limit does not exist, write explicitly DNE. Make sure to describe the method(s) used to obtain the value of the limit.

(a) (5 points) $\lim_{x\to\infty} \dfrac{3x^4 + x - 5}{6x^4 - 2x^2 + 1}$.

Divide coefficient in front of Highest power of x:

$$\lim_{x\to\infty} \frac{3x^4+x-5}{6x^4-2x^2+1} = \frac{3}{6} = \boxed{\frac{1}{2}}.$$

(b) (5 points) $\lim_{x\to-\infty} \dfrac{\sqrt{4x^2+1}}{3x-1}$.

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{3x-1} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{4+1/x^2}}{x(3-1/x)}$$

$$= \lim_{x\to-\infty} \frac{-x}{x} \frac{\sqrt{4+1/x^2}}{3-1/x}$$

$$= \lim_{x\to-\infty} - \frac{\sqrt{4+1/x^2}}{3-1/x}$$

$$= \frac{-\sqrt{4}}{3} = \boxed{-\frac{2}{3}}$$

QUESTION 5 (10 pts)

Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

x: first number
y: second number
P: product

Constraints: $x + 4y = 1000$

2pts. $\Rightarrow$ $x = 1000 - 4y$.

Function: $P = xy = (1000 - 4y)\,y$

$\Rightarrow$ $P(y) = 1000y - 4y^2$. 2pts.

Optimize: $P'(y) = 1000 - 8y$

$= 0$

2pts.

$\Leftrightarrow$ $1000 = 8y$

$\Leftrightarrow$ $y = \dfrac{1000}{8} = \dfrac{250}{2} = 125$

1st test:

2pts.

$y < 125 \Rightarrow 1000 - 8y > 0 \Rightarrow P'(y) > 0$
$y > 125 \Rightarrow 1000 - 8y < 0 \Rightarrow P'(y) < 0$
abs. max at $y = 125$

OR

2nd test: $P''(y) = -8 < 0$ everywhere $\Rightarrow$ abs. max at $y = 125$

Answer:

$\boxed{\begin{array}{l} x = 1000 - 4 \cdot 125 = 500 \text{ 1pt.} \\ y = 125 \text{ 1pt.} \end{array}}$

QUESTION 6 (15 pts)

Answer the following questions.

(a) (5 points) Find the most general antiderivative of $f(x) = 4\sqrt{x} + \cos x - 2\sec^2 x$.

$$F(x) = 4 \cdot \frac{2}{3} x^{3/2} + \sin x - 2\tan x + C$$

$$\Rightarrow \boxed{F(x) = \frac{8}{3} x^{3/2} + \sin x - 2\tan x + C}$$

(b) (5 points) Find $f(x)$ if $f''(x) = 1 - 6x + 48x^2$, $f(0) = 1$ and $f'(0) = 2$.

1st anti-derivative $f'(x) = x - 3x^2 + \frac{48}{3}x^3 + C$

$f'(0) = 2 \Rightarrow C = 2$.

2nd anti-derivative: $f(x) = \frac{x^2}{2} - x^3 + \frac{48}{6}x^4 + 2x + D$

$f(0) = 1 \Rightarrow D = 1$

So, $\boxed{f(x) = 1 + 2x + \frac{x^2}{2} - x^3 + 8x^4}$.

(c) (5 points) Find $f(t)$ if $f'(t) = \dfrac{t^2 + \sqrt{t}}{t}$ and $f(1) = 3$.

$$\frac{t^2 + \sqrt{t}}{t} = t + t^{-1/2} \Rightarrow f(t) = \frac{t^2}{2} + 2t^{1/2} + C$$

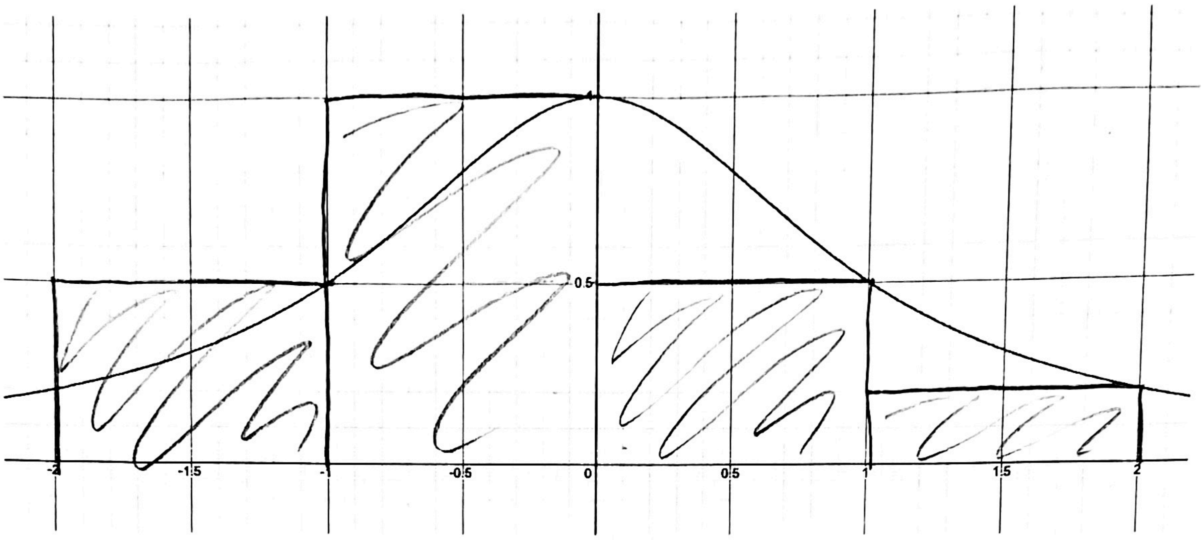
So, $f(1) = 3 \Rightarrow 3 = \frac{1}{2} + 2 + C$

$\Rightarrow \frac{1}{2} = C$

So, $f(t) = \frac{t^2}{2} + 2t^{1/2} + \frac{1}{2}$

QUESTION 7 (10 pts)

The graph of the function $f(x) = \frac{1}{1+x^2}$ is given below.



(a) (5 points) Estimate the area bounded by the graph of $f(x)$ and the x-axis from $a = 0$ to $b = 2$ using two rectangles and right endpoints rule. Is your answer over or under estimating the actual area?

$\Delta x = \frac{2-0}{2} = 1$

$x_1 = 0 + 1 \cdot 1 = 1$

$x_2 = 0 + 2 \cdot 1 = 2$

$h_1 = \frac{1}{2}$

$h_2 = \frac{1}{5}$

$\Rightarrow$ Area $\approx \frac{1}{2} + \frac{1}{5} = \boxed{\frac{7}{10}}$

$\boxed{\text{undu estimating.}}$

(b) (5 points) Estime the area bounded by the graph of $f(x)$ and the x-axis from $a = -2$ to $b = 0$ using two rectangles and the right endpoints rule. Is your answer over or under estimating the actual area?

$\Delta x = \frac{0-(-2)}{2} = 1$

$x_1 = -2 + 1 = -1$

$x_2 = -2 + 2 = 0$

$h_1 = \frac{1}{2}$

$h_2 = 1$

$\Rightarrow$ Area $\approx \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$

$\boxed{\text{overestimating}}$

QUESTION 8 (15 pts)

Answer the following questions.

(a) (5 points) Sketch the region whose area is equal to

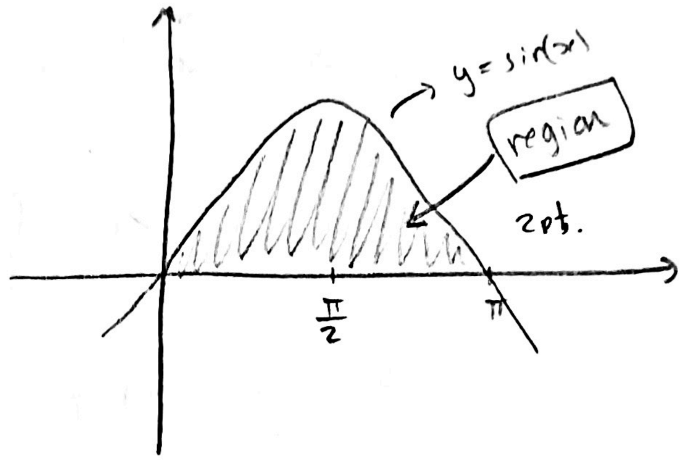
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\frac{i\pi}{n}\right).$$

$\Delta x = \frac{\pi}{n}$ 1pt.

$x_i = \frac{i\pi}{n}$

$x_n = \frac{n\pi}{n} = \pi = b$ 1pt.

$x_0 = \frac{0\pi}{n} = 0 = a$ 1pt.



(b) (5 points) Find the number c satisfying the Mean-Value Theorem with $f(x) = x^2$ on $[0, 2]$.

$f'(x) = 2x$, $a = 0$, $b = 2$

$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$ 2pts.

$\Leftrightarrow 2c = \frac{4 - 0}{2}$ 2pts.

$\Leftrightarrow 2c = 2$

$\Leftrightarrow \boxed{c = 1}$ 1pt.

(c) (5 points) Let $x_1 = -1$. Use Newton's method to find the second approximation x_2 to the root of the equation

$$2x^3 - 3x^2 + 2 = 0.$$

$f(x) = 2x^3 - 3x^2 + 2$ 1pt.

$\Rightarrow f'(x) = 6x^2 - 6x$ 1pt.

So,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = -1 - \frac{2(-1)^3 - 3\cdot 1 + 2}{6\cdot 1 + 6}$$ 2pts.

$$\Rightarrow x_2 = -1 - \left(\frac{-2-3+2}{12}\right)$$

$$\Rightarrow x_2 = -1 + \frac{1}{4}$$

$$\Rightarrow \boxed{x_2 = -\frac{3}{4}}$$ 1pt.

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For official use only:

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	20	10	10	15	10	15	100
Score:	—	—	—	—	—	—	—	—	—