Question 1 (10 pts)

If a spherical snowball melts so that its surface area decreases at a rate of $1\,\text{cm}^2/\text{min}$, find, using **Calculus**, the rate at which the diameter decreases when the diameter is 10 cm.

Note: The surface area of a sphere is $A = 4\pi r^2$.

4 pts for "assembling" info

$$\left\{\begin{aligned} A &= 4\pi r^2 \\ \frac{dA}{dt} &= -1 \text{ cm}^2/\text{min} \end{aligned}\right.$$

r = radius of snowball

D = diameter = 2r

$$A = \pi D^2$$

$$\frac{dA}{dt} = 2\pi D \frac{dD}{dt}$$

1 pt

3 pts

Substitute

$$-1 = 2\pi (10) \frac{dD}{dt}$$

$$\frac{-1}{20\pi} = \frac{dD}{dt}$$

2 pts

QUESTION 2 **(10 pts)**

Let $f(x) = \sqrt{4+x}$.

(a) (5 points) Find the linearization of the function f at the point $a = 0$.

$$f'(x) = \frac{1}{2\sqrt{4+x}} \Rightarrow f'(0) = \frac{1}{4}$$

So,

$$L(x) = \frac{1}{4}(x-0) + f(0)$$

$$\Rightarrow L(x) = \frac{x}{4} + 2$$

(b) (5 points) Using the linearization, estimate the value of $\sqrt{4.1}$. Explain clearly how you obtained your answer and leave it in decimal form.

$$\sqrt{4.1} = \sqrt{4+0.1} \approx L(0.1)$$

So, $$L(0.1) = \frac{0.1}{4} + 2 = 2.025$$

$$\Rightarrow \sqrt{4.1} \approx 2.025$$

Question 3 (22 pts)

Let $f(x) = \dfrac{3x^2 - 3}{x^2 + 3}$.

(a) (4 points) Using **Calculus**, find the vertical asymptotes (if any) and horizontal asymptotes (if any) of the function $f(x)$.

$x^2 + 3 > 0$ for all x, so no vertical asymptotes +1pt

$$\lim_{x\to\infty} \frac{3x^2-3}{x^2+3} = \lim_{x\to\infty} \frac{(3x^2-3)\cdot\frac{1}{x^2}}{(x^2+3)\cdot\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3 - \overbrace{\frac{3}{x^2}}^{\to 0}}{1 + \underbrace{\frac{3}{x^2}}_{\to 0}} = \frac{3}{1} = 3$$

+1pt +1pt

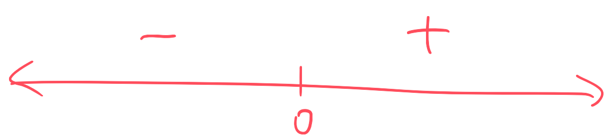
same when $x \to -\infty$, so only horizontal asymptote is

$\boxed{y = 3}$ +1pt

(b) (4 points) The first derivative of f is $f'(x) = \frac{24x}{(x^2+3)^2}$. Find the critical numbers (if any) and the open interval(s) of increase and decrease.

$(x^2+3)^2 > 0$ for all x, so ignore. +1pt

$24x = 0$ when $x = 0$, so $x = 0$ is the only critical number +1pt



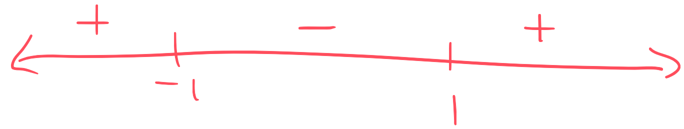
int. of decrease: $(-\infty, 0)$ +1pt

int. of increase: $(0, \infty)$ +1pt

(c) (6 points) The second derivative of f is $f''(x) = \frac{-72(x^2-1)}{(x^2+3)^3}$. Find the x-coordinate of the inflection points (if any) and the open interval(s) of concavity.

$(x^2+3)^3 > 0$ for all x, so ignore +1pt

$-72(x^2-1) = 0$ when $x = \pm 1$. +2pts



if $x < -1$, $-72(x^2-1) > 0$

if $-1 < x < 1$, $-72(x^2-1) < 0$

if $x > 1$, $-72(x^2-1) > 0$

Thus, $x = 1$, $x = -1$ are x-values of inflection points.

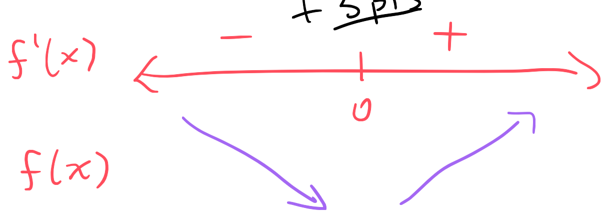
int. of concave up: $(-\infty, -1), (1, \infty)$ +2pts

int. of concave down: $(-1, 1)$ +1pt

(d) (4 points) Using one of the derivative tests, find the local maximum(s) and/or local minimum(s) of the function.

Using First Derivative Test:

+3pts



f must have loc. min at $x = 0$ +1pt

$\therefore$ $\boxed{\text{loc. min is } f(0) = -1.}$

— OR —

Using Second Derivative Test:

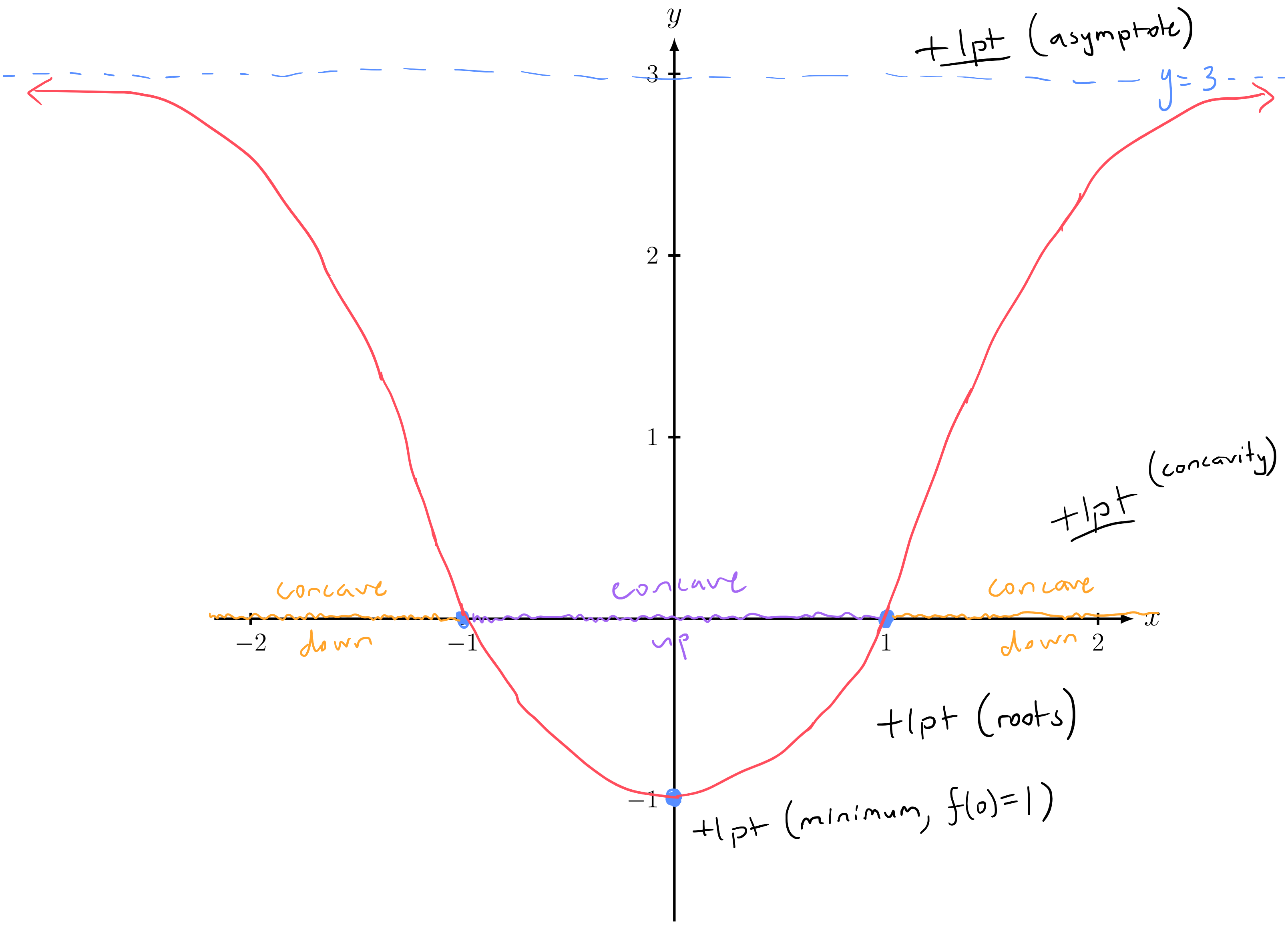
critical number: $x = 0$.

$f''(0) < 0$ implies $f(x)$ has loc. min at $x = 0$. +3pts

$\therefore$ $\boxed{\text{loc. min is } f(0) = -1.}$ +1pt

...Question 3 continued...

(e) (4 points) Sketch the graph of the function f in the axes below. Note that the y-intercept is -1 and the x-intercepts are $x = -1$ and $x = 1$.



QUESTION 4 **(10 pts)**

Compute the following limits. If the limit does not exist, write explicitly DNE. Make sure to write all the details of your calculations.

(a) (5 points) $\lim_{x\to\infty} \frac{3x-2}{2x+1}$.

$$\lim_{x\to\infty} \frac{3x-2}{2x+1} = \lim_{x\to\infty} \frac{x(3-2/x)}{x(2+1/x)}$$ (2 pts)

$$= \lim_{x\to\infty} \frac{3-2/x}{2+1/x}$$ (1 pt)

$$= \frac{3-0}{2+0}$$ (1 pt)

$$= 3/2$$ (1 pt)

No points if just $\frac{3x}{2x} = \frac{3}{2}$ is used. Up to you, though

(b) (5 points) $\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$.

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+1/x^2}}{x(3-5/x)}$$ (1 pt)

$$= \lim_{x\to\infty} \frac{|x|\sqrt{2+1/x^2}}{x(3-5/x)}$$ (1 pt)

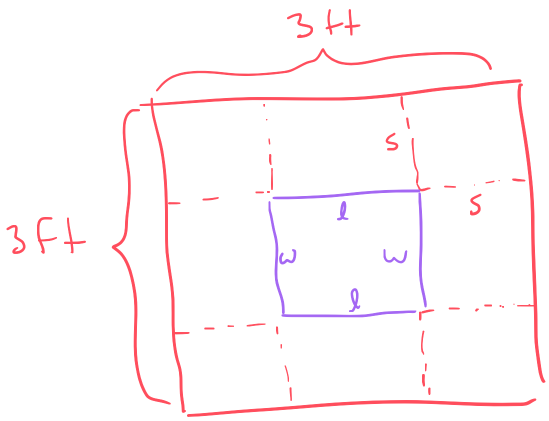
$$= \lim_{x\to\infty} \frac{-x\sqrt{2+1/x^2}}{x(3-5/x)}$$ (1 pt)

$$= \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x}$$ (1 pt)

$$= \frac{-\sqrt{2+0}}{3-0} = -\frac{\sqrt{2}}{3}$$ (1 pt)

QUESTION 5 (15 pts)

A box with an open top is to be constructed from a square piece of cardboard of side length 3 ft by cutting out a square from each of the four corners and bending up the sides. **Using calculus**, find the largest volume that such a box can have. Make sure to justify clearly your answer.



$+1$ pt (picture with labels)

$V = l \cdot w \cdot h$ $+1$ pt (volume formula)

$l = 3-2s$

$w = 3-2s$ $+2$ pts (variables in terms of s)

$h = s$

$V(s) = (3-2s)(3-2s)s$ $+1$ pt (substitution in V formula)

$= 4s^3 - 12s^2 + 9s$

$V'(s) = 12s^2 - 24s + 9$ $+2$ pts (derivative)

$0 = 12s^2 - 24s + 9$

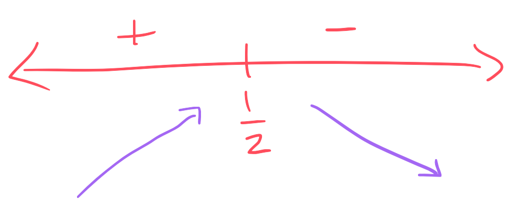
quadratic formula or factor to get $+2$ pts (solving technique)

$s = \frac{1}{2}$ and $s = \frac{3}{2}$ $+1$ pt (roots)

outside of domain of problem (only want $0 < s < \frac{3}{2}$) $+1$ pt

$V(0) = 9 > 0$

$V(1) = 12 - 24 + 9 < 0$



...check it's a maximum $+2$ pts

$\therefore$ Largest volume is $V(\frac{1}{2}) = 2\text{ ft}^3$ $+1$ pt (units!)

$+1$ pt (finding $V(\frac{1}{2})$)

QUESTION 6 (15 pts)

Find the most general antiderivative of the following functions.

(a) (5 points) $f(x) = 4\sqrt{x} - 6x^2 + 3$.

$$\therefore \int f(x)\,dx = \int (4\sqrt{x} - 6x^2 + 3)\,dx$$

$$= 4\int x^{1/2}dx - 6\int x^2 dx + 3\cdot\int 1\cdot dx \Big\}\ 1\text{-pt}$$

$$= 4\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 6\frac{x^{2+1}}{2+1} + 3x + c \Big\}\ \text{each term 1-pt}$$

$$= \boxed{\frac{8}{3}x^{3/2} - 2x^3 + 3x + c} \leftarrow \text{1 pt (final answer + "c")}$$

(b) (5 points) $f(x) = \cos(x) + 2\sec^2(x)$.

$$\therefore \int f(x)\,dx = \int \cos(x) + 2\sec^2(x)\,dx$$

$$= \int \cos(x)\,dx + 2\int \sec^2(x)\,dx \Big\}\ \text{2 pts}$$

$$= \boxed{\sin(x) + 2\tan(x) + c} \leftarrow \text{2 pt correct expression, 1 pt for the "+c"}$$

1pt 1pt 1pt

(c) (5 points) $f(x) = x\sqrt{x} + \dfrac{x^2+x}{x}$.

$$= x\sqrt{x} + x + 1 \Big\}\ \text{1 pt for algebraic simplification}$$

$$\therefore \int f(x)\,dx = \int (x\sqrt{x} + x + 1)\,dx$$

$$= \int x\cdot x^{1/2}dx + \int x\,dx + \int 1\,dx \Big\}\ \text{1 point for separation}$$

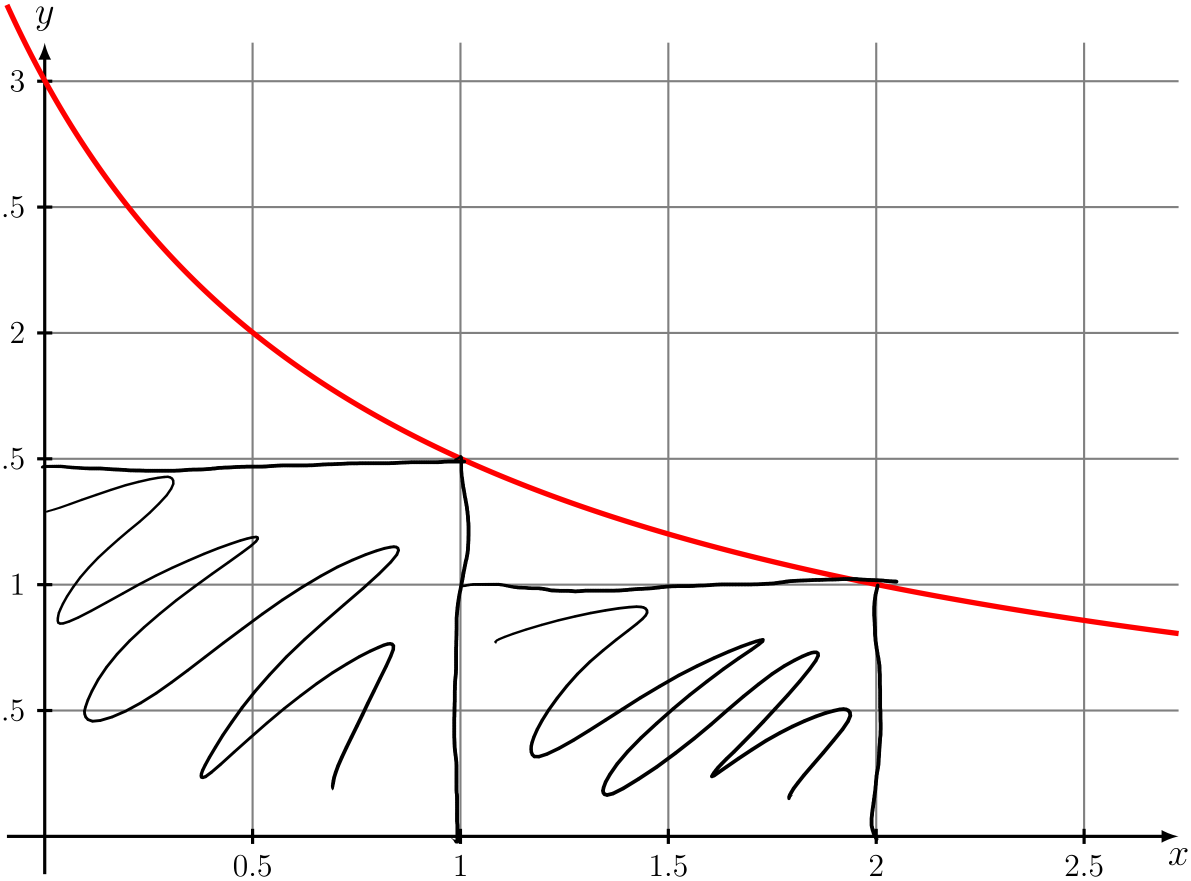
$$= \int x^{3/2}dx + \frac{1}{2}x^2 + x + c$$

$$= \frac{x^{3/2+1}}{3/2+1} + \frac{1}{2}x^2 + x + c = \boxed{\frac{2}{5}x^{5/2} + \frac{1}{2}x^2 + x + c}$$

1pt 1pt 1pt

QUESTION 7 **(8 pts)**

The graph of the function $f(x) = \dfrac{3}{1+x}$ is given below.



(a) (4 points) Estimate the area under the curve from $a = 0$ to $b = 2$ using two rectangles and right endpoints.

$n = 2$ $\Rightarrow$ Area $\approx 1 \cdot 1.5 + 1 \cdot 1$

$\Delta x = \dfrac{2-0}{2} = 1$ $= 2.5$ or $\dfrac{5}{2}$.

(b) (2 points) Draw the two rectangles from part (a) on the above picture of the graph of $f(x)$.

(c) (2 points) Is your answer over or under approximating the actual value of the area under the curve?

Under

QUESTION 8 **(10 pts)**

Below is the recordings of the speed of a Kangoroo in Australia.

Time (s)	0	5	10	15	20	25	30	35	40
Speed (ft/s)	6	3	2	3	4	7	5	1	4

(a) (5 points) Using only the recordings at every 10 seconds, estimate the distance travelled after 40 seconds.

R

Distance $\approx 10 \cdot 2 + 10 \cdot 4 + 10 \cdot 5 + 10 \cdot 4$

$= 20 + 40 + 50 + 40 = 150$ feet.

L

Distance $\approx 0 \cdot 10 + 2 \cdot 10 + 4 \cdot 10 + 5 \cdot 10$

$= 20 + 40 + 50$

$= 110$ feet.

(b) (5 points) Does your estimate get better if you use instead the recordings at every 5 seconds?

Yes because you increase the precision.

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Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	22	10	15	15	8	10	100
Score:									