Solutions Midterm 02 (Sample).

Question 1

(1) $A = 4\pi r^2 \Rightarrow r = \sqrt{\frac{A}{4\pi}} \Rightarrow r = \sqrt{\frac{10}{4\pi}} = \sqrt{\frac{5}{2\pi}}$

Now,

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \Rightarrow -2 = 8\pi \sqrt{\frac{5}{2\pi}} \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = -\frac{1}{4\pi}\sqrt{\frac{2\pi}{5}}$$

(2) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\Rightarrow \frac{dV}{dt} = \cancel{4\pi}\left(\frac{5}{2\pi}\right) \cdot -\frac{1}{\cancel{4\pi}}\left(\frac{2\pi}{5}\right)^{1/2}$$

$$\Rightarrow \boxed{\frac{dV}{dt} = -\sqrt{\frac{5}{2\pi}} \ \text{cm}^3/\text{min}}$$

Question 2

$\sqrt{x+2}$ becomes $\sqrt{x+4}$. $\sqrt{2.1}$ becomes $\sqrt{4.1}$

(a) $f'(x) = \dfrac{1}{2\sqrt{x+4}} \cdot (x+4)' = \dfrac{1}{2\sqrt{x+4}} \cdot (1) = \dfrac{1}{2\sqrt{x+4}}$

So, $f'(0) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$ & $f(0) = 2$

$\Rightarrow \quad L(x) = f'(0)(x-0) + f(0)$

$= \boxed{\dfrac{x}{4} + 2}$

(b) $\sqrt{4.1} \approx L(0.1) = \dfrac{0.1}{4} + 2 = \boxed{2.025}$

Question 3.

(a) $f'(x) = 3x^2 - 1 \Rightarrow x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$\Rightarrow x_2 = 1 - \frac{(1^3 - 1 + 1)}{(3(1)^2 - 1)}$$

$$\Rightarrow x_2 = 1 - \frac{1}{2} \Rightarrow \boxed{x_2 = \frac{1}{2}}$$

(b) When $f'(x_1) = 0$.

$$\Leftrightarrow 3x_1^2 - 1 = 0$$

$$\Leftrightarrow x_1^2 = \frac{1}{3}$$

$$\Leftrightarrow \boxed{x_1 = \frac{1}{\sqrt{3}} \quad \text{or} \quad x_1 = -\frac{1}{\sqrt{3}}}$$

Question 4

Domain: $(-\infty, 1) \cup (1, \infty)$.

Vertical Asymptotes:
$$\lim_{x \to 1^-} \frac{x}{x-1} = \frac{1}{1^- - 1} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to 1^+} \frac{x}{x-1} = \frac{1}{1^+ - 1} = \frac{1}{0^+} = +\infty$$

Horizontal Asymptotes:
$$\lim_{x \to \infty} \frac{x}{x-1} = 1 \quad \& \quad \lim_{x \to -\infty} \frac{x}{x-1} = 1$$

$y = 1$ is an HA.

Symmetries: $f(x)$ is not odd and not even.

Derivative:
$$f'(x) = \frac{x'(x-1) - x(x-1)'}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = -\frac{1}{(x-1)^2}$$

C.N. is $x = 1$ because $f'(x)$ DNE at $x = 1$.

Since $f'(x) = \frac{-1}{(x-1)^2} < 0$ for all $x \neq 1$, then

f is decreasing everywhere on its domain.

<u>Second derivative</u>: $f''(x) = \left(-\frac{1}{(x-1)^2}\right)' = -\left((x-1)^{-2}\right)'$

$\Rightarrow f''(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$

If $x<1$, $\quad x-1<0 \quad \Rightarrow f''(x) < 0, \quad x<1$

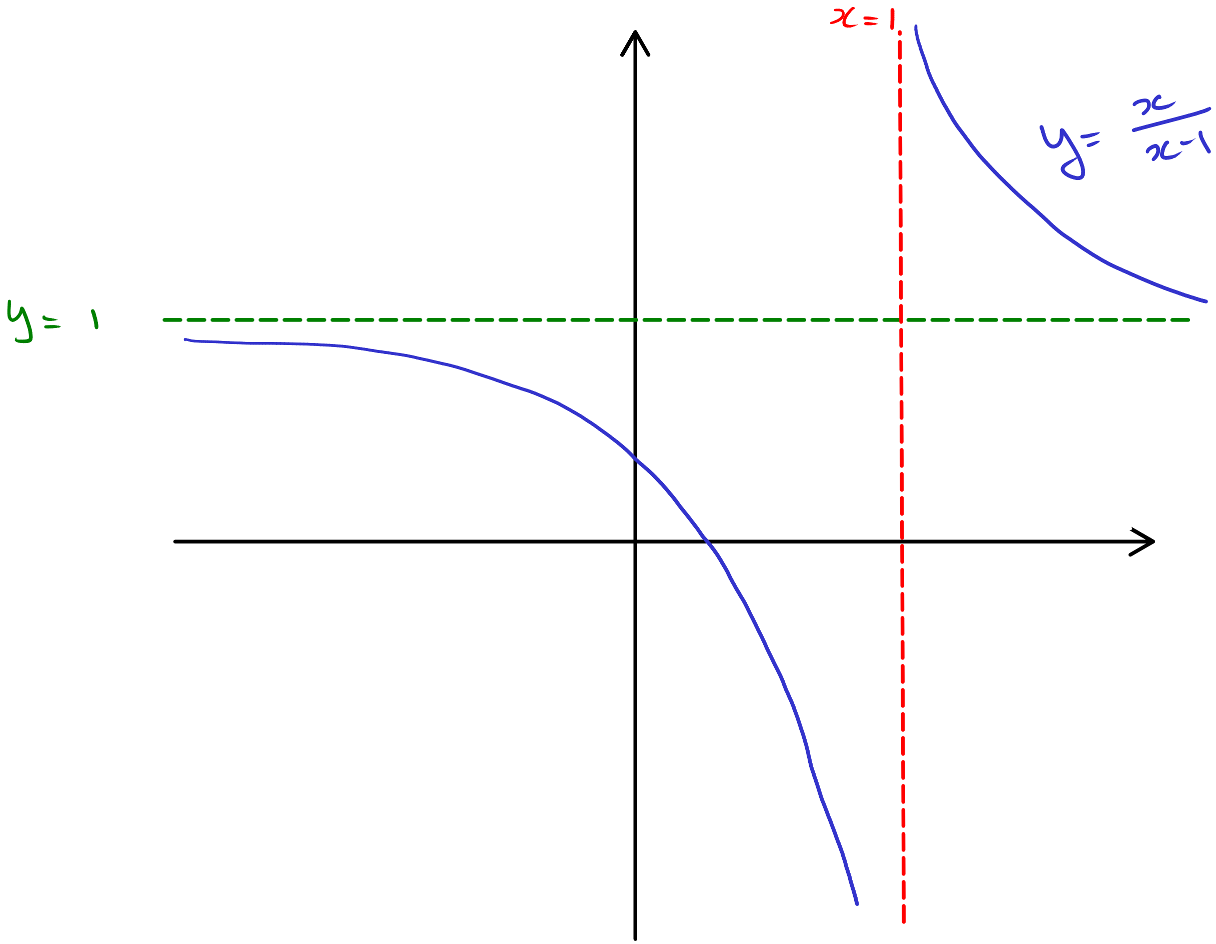
$\Rightarrow f$ concave down on $(-\infty, 1)$

If $x>1$, $\quad x-1>0 \quad \Rightarrow f''(x) > 0, \quad x>1$

$\Rightarrow f$ concave up on $(1, \infty)$.

No max & no min ...

Graph:



Question 5.

(a) $$\lim_{x\to\infty} \frac{x^3+4x+2}{10x^3+x^2+10} = \boxed{\frac{1}{10}}$$

(b) $$\lim_{x\to\infty} \frac{x+4}{\sqrt[3]{x^3+x+3}} = \lim_{x\to\infty} \frac{x(1+4/x)}{\sqrt[3]{x^3}\sqrt[3]{1+1/x^2+3/x^3}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}(1+4/x)}{\cancel{x}\sqrt[3]{1+1/x^2+3/x^3}}$$

$$= \lim_{x\to\infty} \frac{1+4/x}{\sqrt[3]{1+1/x^2+3/x^3}}$$

$$= \frac{1+0}{\sqrt[3]{1+0+0}} = \boxed{1}$$

(c) $$\lim_{x\to-\infty} \frac{\sqrt{2x^2+2}}{x+4} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{2+2/x^2}}{x(1+4/x)}$$

$$= \lim_{x\to-\infty} \frac{-x}{x} \frac{\sqrt{2+2/x^2}}{1+4/x} = \boxed{-\frac{\sqrt{2}}{1}}$$

Question 6.

(a) $\sum_{i=1}^{5} 2i = 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5$

$= 2+4+6+8+10 = 30$

(shortcut: $\sum_{i=1}^{5} 2i = 2(1+2+3+4+5) = 2\sum_{i=1}^{5} i$

$= 2\cdot\left(\frac{5\cdot 6}{2}\right) = 30.$

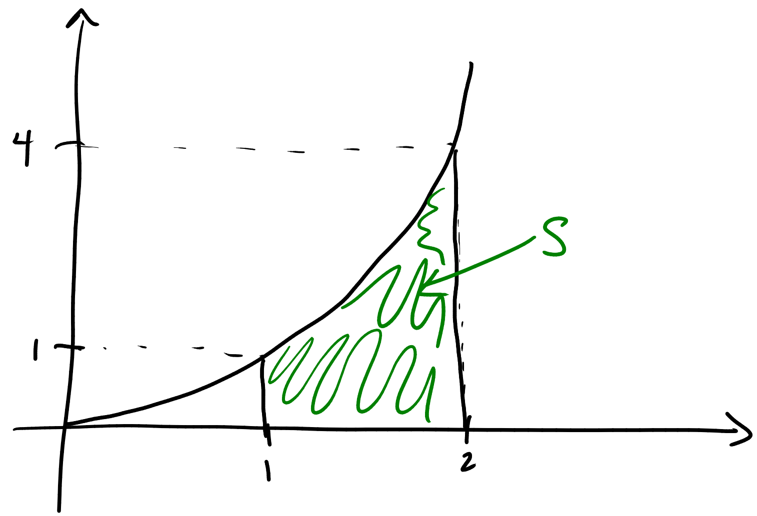
(b) $\Delta x = \frac{1}{n}$

$x_i = 1 + \frac{i}{n}$ (Right endpoints)

$a = x_0 = 1$

$b = x_n = 1 + \frac{n}{n} = 2$

$f(x) = x^2$



(c) $f'(x) = \frac{-1}{x^2}$. So, find c s.t.

$$f'(c) = \frac{f(3) - f(1)}{3-1}$$

$$\iff \frac{-1}{c^2} = \frac{\frac{1}{3} - 1}{2}$$

$$\iff \frac{-1}{c^2} = \frac{-2/3}{2} = \frac{-1}{3}$$

$$\iff 3 = c^2$$

$$\iff c = \pm\sqrt{3}$$

But c must be in [1, 3] $\Rightarrow$ $\boxed{c = \sqrt{3}}$ only

Question 7.

x: length of base (meters)
y: length of heigth (meters)
A: area (m^2)
P: perimeter (m).

$$A = xy \quad \text{and} \quad P = 2x + 2y.$$

So, $A = 1000 \implies 1000 = xy \implies y = \dfrac{1000}{x}$

So, $P = 2x + \dfrac{2000}{x}$.

Derivative: $P' = 2 - \dfrac{2000}{x^2} = 0 \iff x^2 = 1000 \iff x = \sqrt{1000}$

If $x < \sqrt{1000}$, $P'(x) < 0$ & $x > \sqrt{1000}$, $P'(x) > 0$, then <u>min</u>.

Answer: $x = \sqrt{1000}$ m & $y = \dfrac{1000}{\sqrt{1000}} = \sqrt{1000}$ m.

Question 8

(a) $\Delta x = \dfrac{\pi/2 - 0}{2} = \dfrac{\pi}{4}.$

$a = 0$

$b = \dfrac{\pi}{2}$

$\Rightarrow$

$x_1 = 0 + \dfrac{\pi}{4} = \dfrac{\pi}{4} \rightarrow h_1 = \sin(\pi/4) = \dfrac{\sqrt{2}}{2}$

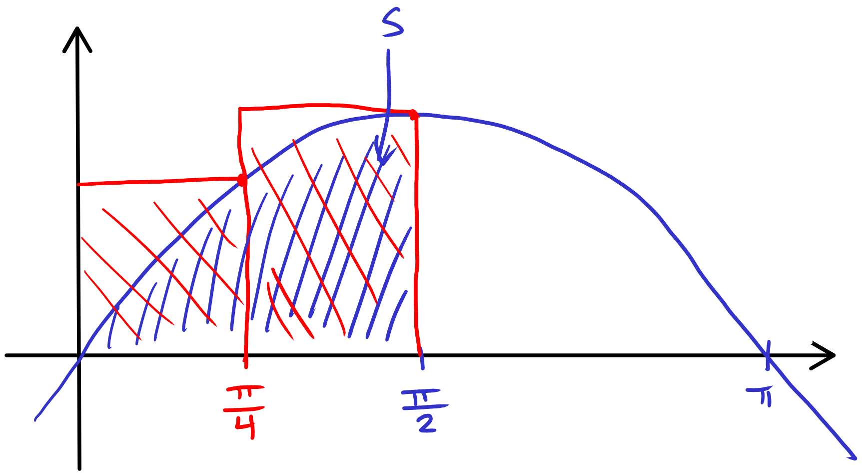
$x_2 = 0 + 2 \cdot \dfrac{\pi}{4} = \dfrac{\pi}{2} \rightarrow h_2 = \sin(\pi/2). = 1$

So,

$$\text{Area} \approx \Delta x \cdot h_1 + \Delta x \cdot h_2 = \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\pi}{4} \cdot 1$$

$$\Rightarrow \text{Area} \approx \frac{\sqrt{2}\,\pi}{8} + \frac{\pi}{4}.$$

(b)



Note: You can also use the left endpoints of the subintervals.