Chapter 1

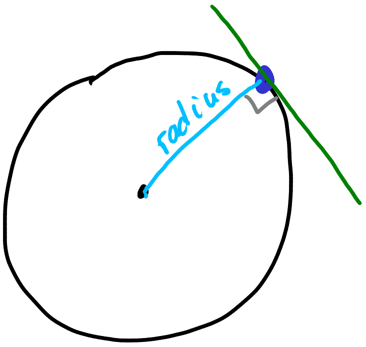
Functions and Limits

1.4 The Tangent and Velocity Problems

The Tangent problem.

Example. What is the tangent to a circle?

Illustration: https://www.desmos.com/calculator/7qflpgcuay

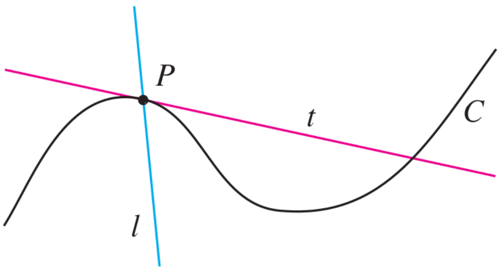


In Geometry, a TANGENT LINE at a given point on a a circle is a line that touches the circle only at that point.

Problems with this definition:

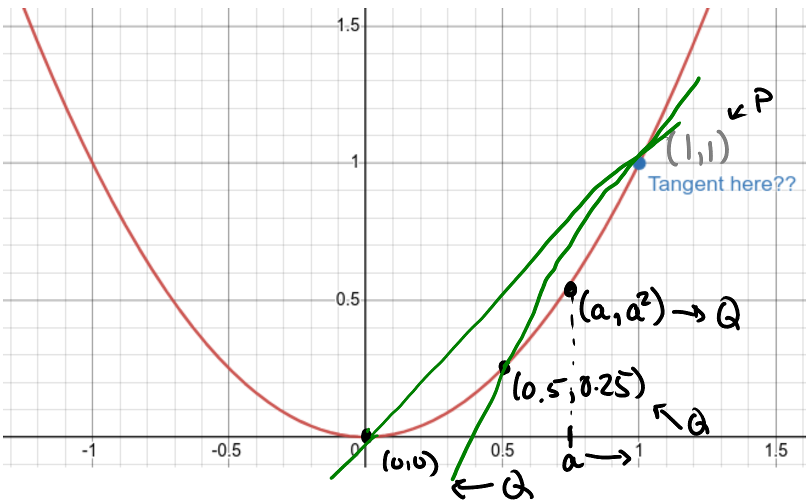
1) Not all curves are circle!

2) For other curves, the tangent line may intersect at several points!



EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Go play around with this problem: https://www.desmos.com/calculator/kbfn4ptdop



① Find slope secant line.

$$Q(0,0): \quad m_{PQ} = \frac{y_P - y_Q}{x_P - x_Q} = \frac{1-0}{1-0} = 1$$

$$Q(0.5, 0.25): \quad m_{PQ} = \frac{y_P - y_Q}{x_P - x_Q} = \frac{1-0.25}{1-0.5} = 1.5$$

Take an arbitrary point (a, a^2) on the parabola:

$$m_{PQ} = \frac{1-a^2}{1-a}$$

Make $a \longrightarrow 1$?

$$m = \lim_{a \to 1} \frac{1-a^2}{1-a} = 2$$

② Find Eq. of tangent line

$m = 2$

$(x_0, y_0) = (1,1)$

$$y - y_0 = m(x - x_0) \rightarrow y - 1 = 2(x-1)$$

$$\rightarrow \boxed{y = 2x - 1}$$

Main concept: The SLOPE of the tangent line is the LIMIT of the slopes of the secant lines.

The Velocity Problem.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

$\boxed{\text{Galileo: } s(t) = \underline{4.9}t^2}$

$s'(t) = 2 \cdot 4.9 \cdot t$

Average velocity: $\dfrac{\text{variation in position}}{\text{variation in time}}$

$$\Rightarrow \quad V_{av} = \frac{s(t) - s(5)}{t - 5}$$

t	V_{av}
0	24.5
4	44.1
4.5	46.5
4.9	48.61...
4.99	48.951...
4.999	48.9951...
$\downarrow$	$\downarrow$
5	49

Conclusion

as $t \to 5$

$v_{av} \longrightarrow 49$

$\downarrow$

$\dfrac{s(t)-s(5)}{t-5}$

$$\text{velocity at 5 seconds} = \lim_{t\to 5} \frac{s(t)-s(5)}{t-5} = 49\,m/s.$$

$\downarrow$

$s'(5)$

derivative.

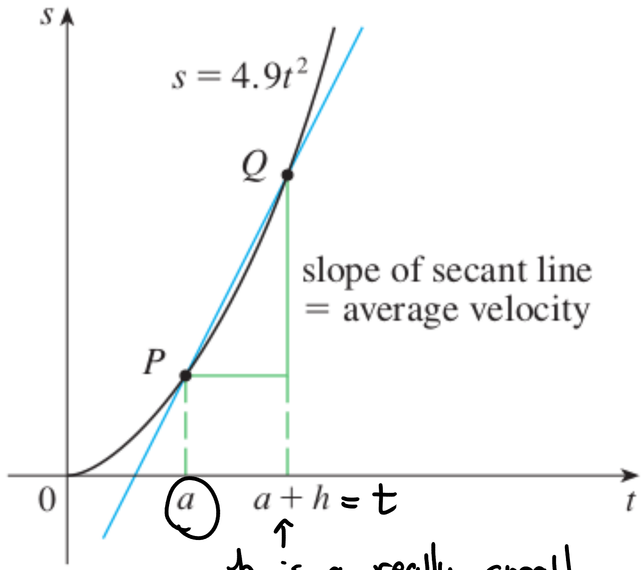
Illustration on Desmos: https://www.desmos.com/calculator/gxry2fylvw

Average velocity.

$$v_{av} = \frac{s(t)-s(\overset{5}{\overset{\downarrow}{a}})}{t-\underset{\underset{5}{\uparrow}}{a}}$$

↳ slope of secant passing through P & Q.

Relation to the secant line.



h is a really small number

Instantaneous Velocity.

Made $t \to a$

$v_{av} \longrightarrow$ inst. velocity

slope of the tangent line passing through P

Relation to the tangent line.