Chapter 1

Functions and Limits

1.5 The Limit of a Function

Intuitive definition of a limit.

$m_{PQ} = \dfrac{1-x^2}{1-x} = f(x)$

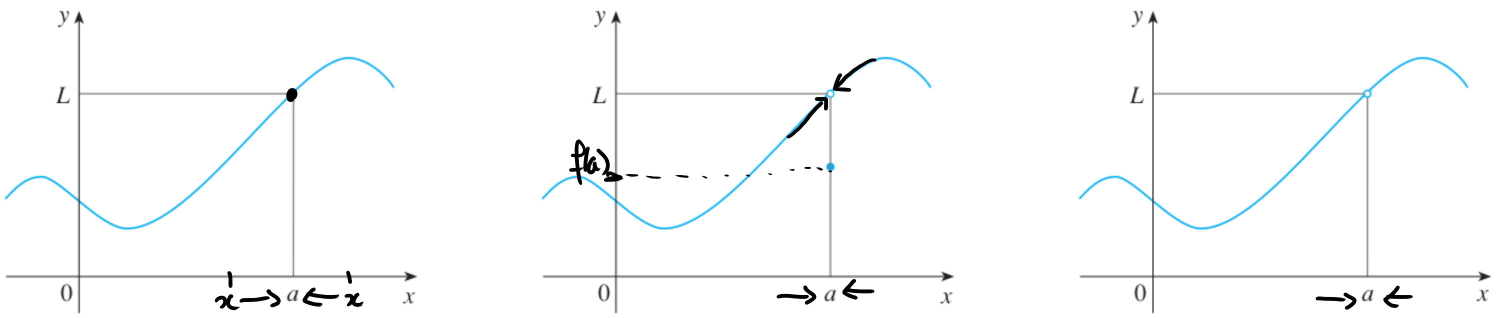
1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Three cases:



(a) $f(a)$ define & $L = f(a)$

(b) $L \neq f(a)$

(c) $f(a)$ is not define.

EXAMPLE 1 Guess the value of $\displaystyle\lim_{x \to 1} \frac{x-1}{x^2-1}$.

$f(x) = \dfrac{x-1}{x^2-1}$ & $a = 1$

x	$f(x)$
0.8	0.5556
0.9	0.52632
0.99	0.50251
0.999	0.50025

$\downarrow$ 0.5

x	$f(x)$
1.2	0.45454
1.1	0.47619
1.01	0.49751
1.001	0.49975

$x \downarrow 1$, $f(x) \downarrow 0.5$

Our guess: $\displaystyle\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$

EXAMPLE 3 Guess the value of $\lim_{x \to 0} \underbrace{\frac{\sin x}{x}}$.

$f(x) = \frac{\sin x}{x}$, $a = 0$.

Check Desmos. https://www.desmos.com/calculator/5tdinsunzj

Guess: $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

EXAMPLE 4 Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$. ← $f(x)$, a

x	$f(x)$
0.5	0
0.1	0
0.01	0
0.001	0
↓	↓
0	0

	x	$f(x)$
$k=1$	$2/5$	1
$k=2$	$2/9$	1
$k=3$	$2/13$	1
$k=4$	$2/17$	1
	↓	↓
	0	1

$\sin \frac{\pi}{0.5} = \sin 2\pi = 0$

$\sin\left(\frac{\pi}{x}\right) = 1 \iff \frac{\pi}{x} = \frac{\pi}{2} + 2k\pi$

$\iff \frac{\cancel{\pi}}{x} = \frac{(4k+1)\cancel{\pi}}{2}$

$\iff x = \frac{2}{4k+1}$

$\lim_{x \to 0} \sin \frac{\pi}{x} \quad \nexists$

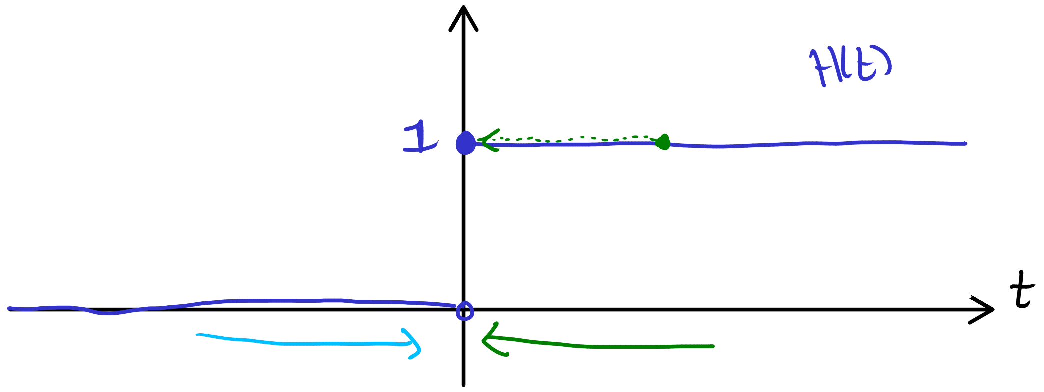
$\boxed{\text{Fund Prop. limit is Unique !}}$

One-sided Limits.

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

What is the limit when t approached 0 from the right and when t approaches 0 from the left.



* From the right.

$H(t) \longrightarrow 1$ when $t \longrightarrow 0$ (right) $t > 0$

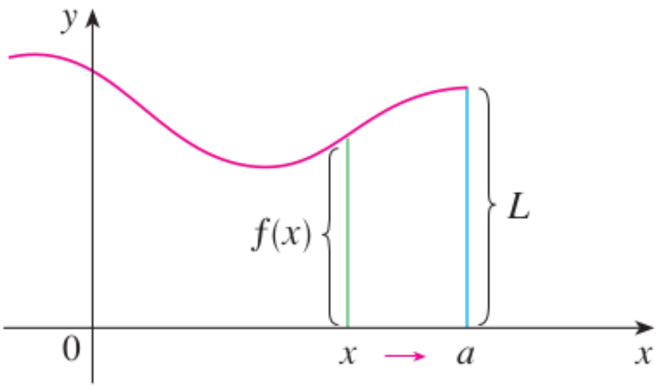
$$\lim_{t \to 0^+} H(t) = 1$$

* From the left.

$H(t) \longrightarrow 0$ when $t \to 0$ (left). $(t < 0)$

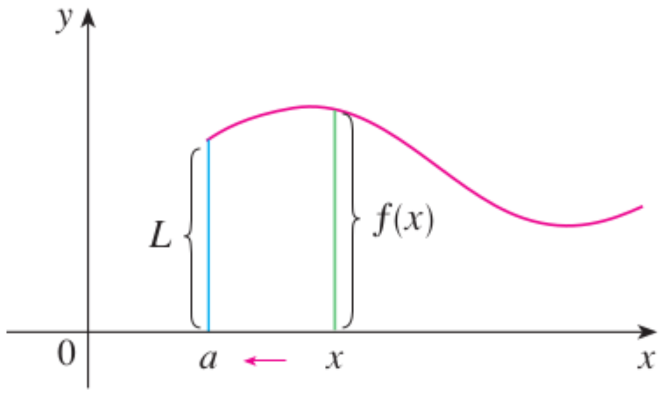
$$\lim_{t \to 0^-} H(t) = 0$$

Left-hand limits.



(a) $\lim_{x\to a^-} f(x) = L$

Right-hand limits.



(b) $\lim_{x\to a^+} f(x) = L$

Fundamental Property:

$\lim_{t\to 0} H(t) \nexists$

$$\boxed{3} \quad \lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x\to 2^-} g(x) = 3$ (b) $\lim_{x\to 2^+} g(x) = 1$ (c) $\lim_{x\to 2} g(x)$ $\nexists$

(d) $\lim_{x\to 5^-} g(x) = 2$ (e) $\lim_{x\to 5^+} g(x) = 2$ (f) $\lim_{x\to 5} g(x) = 2$

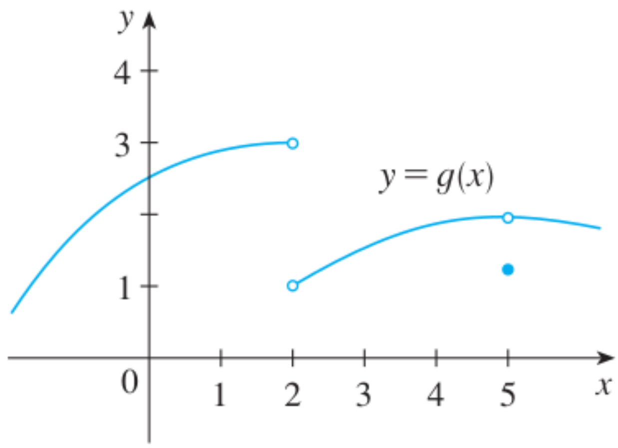


Figure 10.

Infinite limits.

EXAMPLE 8 Find $\lim_{x\to 0} \frac{1}{x^2}$ if it exists.

x	$f(x)$
0.5	4
−0.1	100
0.01	10000
−0.001	1000000
↓	↓
0	∞

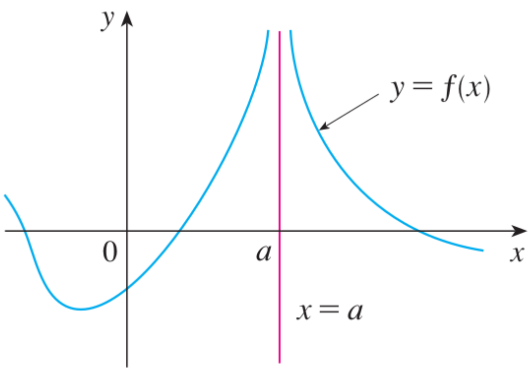
$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Positive infinity.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

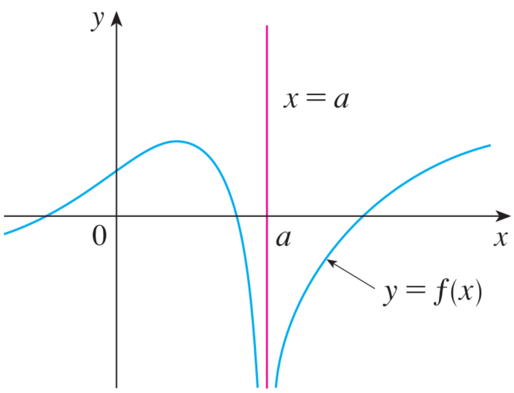


Negative Infinity

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

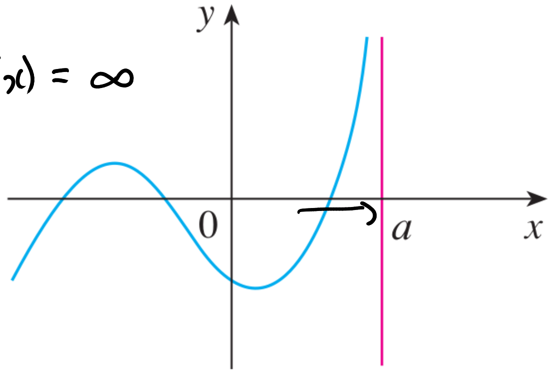
means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.



Other types of infinite limits.

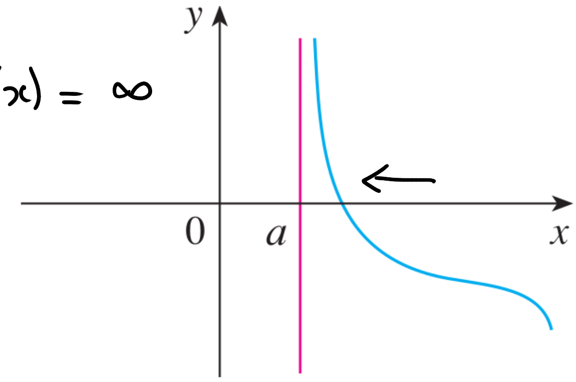
a)

$\lim_{x \to a^-} f(x) = \infty$

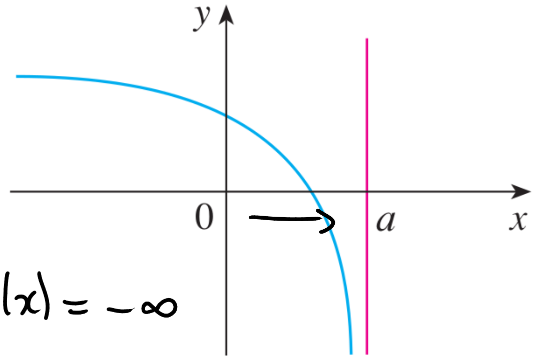


b)

$\lim_{x \to a^+} f(x) = \infty$

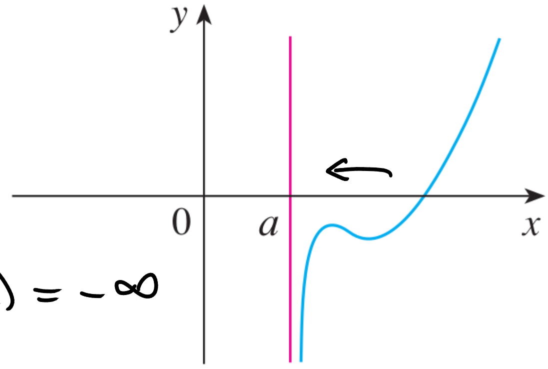


c)



$\lim_{x \to a^-} f(x) = -\infty$

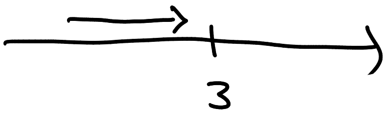
d)



$\lim_{x \to a^+} f(x) = -\infty$

EXAMPLE 9 Find $\lim_{x \to 3^+} \dfrac{2x}{x-3}$ and $\lim_{x \to 3^-} \dfrac{2x}{x-3}$.

1)

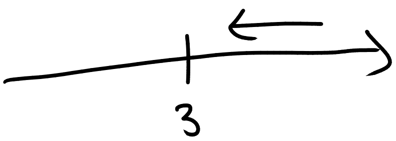


$2x \longrightarrow 2 \cdot 3 = 6$

$x - 3 \longrightarrow 0^-$

$$\Rightarrow \lim_{x \to 3^-} \frac{2x}{x-3} = \frac{6}{0^-} = -\infty$$

2)



$2x \longrightarrow 6^+$

$x - 3 \longrightarrow 0^+$

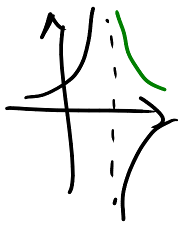
$$\Rightarrow \lim_{x \to 3^+} \frac{2x}{x-3} = \frac{6}{0^+} = +\infty$$

Valid if you don't have $\boxed{\dfrac{0}{0}}$

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

$$\tan x = \frac{\sin x}{\cos x} \rightarrow \cos x = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi$$

k is int.

Test $x = \frac{\pi}{2}$

$$\lim_{x \to (\frac{\pi}{2})^-} \tan x = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sin x}{\cos x}$$

$$= \frac{1}{0^+} = +\infty$$

$$\lim_{x \to (\frac{\pi}{2})^+} \tan x = \frac{1}{0^-} = -\infty.$$