$$\lim_{x \to a} f(x) = L$$

Chapter 1

Functions and Limits

1.6 Calculating Limits Using the Limit Laws

EXAMPLE 1 Use the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x\to -2} [f(x) + 5g(x)]$ (b) $\lim_{x\to 2}[f(x)g(x)]$ (c) $\lim_{x\to -2} \frac{f(x)}{g(x)}$

(d) $\lim_{x\to -2}[2f(x)] = 2$ (e) $\lim_{x\to -2}[f(x) - g(x)]$

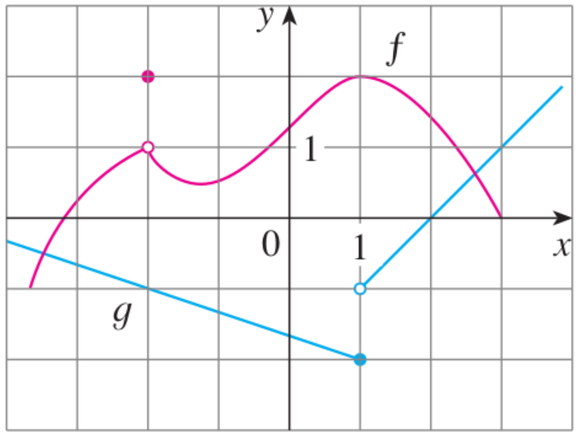

FIGURE 1

Use Desmos
https://www.desmos.com/calculator/7fy0x0ghia

(a) $\lim_{x\to -2} [f(x)+5g(x)] = -4$

$= 1 + 5(-1)$

$= \lim_{x\to -2} f(x) + 5 \lim_{x\to -2} g(x)$

(b) $\lim_{x\to 2} f(x)g(x) = 0$

$= 1.4 \cdot 0$

$= \lim_{x\to 2} f(x) \cdot \lim_{x\to 2} g(x)$.

(c) $\lim_{x\to -2} \frac{f(x)}{g(x)} = \frac{\lim_{x\to -2} f(x)}{\lim_{x\to -2} g(x)} = \frac{1}{-1} = -1$

(e) $\lim_{x\to -2} [f(x) - g(x)] = \lim_{x\to -2} f(x) + \lim_{x\to -2} [-g(x)]$

$= 1 - \lim_{x\to -2} g(x)$

$= 1 - (-1) = \boxed{2}$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$.
4. $\lim_{x\to a} [f(x)\,g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$.
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

EXAMPLE. Think of two ways of computing the following limit:

$$\lim_{x\to 2}(1+x)^3$$

$$\lim_{x\to 2}(1+x)^3 = \lim_{x\to 2}\left[(1+x)\right]\left[(1+x)(1+x)\right]$$

$$= \lim_{x\to 2} 1+x \lim_{x\to 2}\left[(1+x)\right]\left[(1+x)\right]$$

$$= \lim_{x\to 2} 1+x \lim_{x\to 2} 1+x \lim_{x\to 2} 1+x$$

$$= \left(\lim_{x\to 2} 1+x\right)^3 = \left(\lim_{x\to 2} 1 + \lim_{x\to 2} x\right)^3$$

$$= (1+2)^3 = \boxed{27}$$

EXAMPLE. Think of two ways of computing the following limit:

$$\lim_{x\to\pi/4}\cos^2(x)$$

$$\lim_{x\to\pi/4}\cos^2 x = \left(\lim_{x\to\pi/4}\cos x\right)\left(\lim_{x\to\pi/4}\cos x\right) \quad \text{[Prod. Rule]}$$

$$= \left(\lim_{x\to\pi/4}\cos x\right)^2 = \left(\cos \pi/4\right)^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{1}{2}}$$

General Formula:

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

Special cases:

$$\lim_{x\to a} 1 = 1, \quad \lim_{x\to a} x^n = a^n$$

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4) = L$ (b) $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = L$

(a) $L = \lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4$ [Sum & Diff. Rules]

$= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + 4 \lim_{x \to 5} 1$ [Const. Rule]

$= 2 \cdot 5^2 - 3 \cdot 5 + 4$

$= 39$

(b) ① $\lim_{x \to -2} 5 - 3x = \lim_{x \to -2} 5 - 3 \lim_{x \to -2} x = 5 - 3(-2)$

$= 11 \neq 0$

Quotient Rule:

$$L = \frac{\lim_{x \to -2} x^3 + 2x^2 - 1}{\lim_{x \to -2} 5 - 3x} = \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{11}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{11}$$

$$= \frac{-1}{11}$$

Remark:

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Root Law.

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

Example. Compute $\lim_{u\to -2} \sqrt[2]{u^4 + 3u + 6}$.

$x^2 = 16$

$\sqrt{x^2} = \sqrt{16}$

$\hookrightarrow \pm x = 4$

$\lim_{u\to -2} (u^4 + 3u + 6) = 16 > 0$

$\Rightarrow \lim_{u\to -2} \sqrt{u^4 + 3u + 6} = \sqrt{16}$

$= 4$

EXAMPLE 3 Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$. $\rightarrow$ $\frac{0}{0}$ not defined !!.

can't subs. Rule or Quotient.

$$\frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1 \quad (x \neq 1)$$

So,

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} x+1 = 2$$

We have to use the following new substitution rule:

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

EXAMPLE 5 Evaluate $\lim_{h\to 0} \frac{(3+h)^2 - 9}{h}$. $\longrightarrow f(h)$

$$\frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h}$$

$$= \frac{6h + h^2}{h}$$

$$= \frac{(6+h)\cancel{h}}{\cancel{h}} \quad (h \neq 0)$$

$$= 6 + h \quad (h \neq 0)$$

$$\lim_{h\to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h\to 0} 6 + h = \boxed{6}$$

EXAMPLE 6 Find $\lim_{t \to 0} \underbrace{\frac{\sqrt{t^2+9}-3}{t^2}}_{f(t)}$. $\to \frac{0}{0}$ Undefined.

Simplify:

$$f(t) = \frac{(\sqrt{t^2+9}-3)}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

$$= \frac{t^2+\cancel{9}-\cancel{9}}{t^2(\sqrt{t^2+9}+3)}$$

$$= \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)} \quad (t \neq 0)$$

$$= \frac{1}{\sqrt{t^2+9}+3}$$

$$\lim_{t \to 0} \sqrt{t^2+9}+3 = \sqrt{9}+3 = 6 \neq 0$$

So,

$$L = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3} \overset{\text{Quot. Law}}{=} \frac{1}{6}$$

EXAMPLE 8 Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

REMARK: ALL LIMIT RULES WORK FOR LIMITS FROM THE LEFT AND FROM THE RIGHT.

$$|-2| = -(-2)$$

1) $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-\cancel{x}}{\cancel{x}} = \lim_{x \to 0^-} -1 = -1$

2) $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{\cancel{x}}{\cancel{x}} = \lim_{x \to 0^+} 1 = 1$

$\Rightarrow \lim_{x \to 0} \frac{|x|}{x} \;\nexists$.

$\exists$: exist

$\nexists$: not exist.

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

1) $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 8 - 2x = 8 - 2 \cdot 4 = 0$

2) $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{\lim_{x \to 4^+} x - 4} = 0$

So, $\boxed{\lim_{x \to 4} f(x) = 0}$

EXAMPLE 11 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

$\lim_{x\to 0} \sin\frac{\pi}{x} \nexists$

Use prod. Rule.

$\underbrace{\lim_{x\to 0} x^2}_{0} \underbrace{\lim_{x\to 0} \sin\frac{1}{x}}_{\nexists} \longrightarrow$ Product useless.

Way to do it.

$$-1 \le \sin A \le 1$$

$A = \frac{1}{x} \quad (x \neq 0) \Rightarrow \quad -1 \le \sin\frac{1}{x} \le 1$

$$\Rightarrow \quad -x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$\lim_{x\to 0} x^2 = 0 \quad \& \quad \lim_{x\to 0}(-x^2) = 0$

this forces

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

3 **The Squeeze Theorem** If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$