Chapter 1

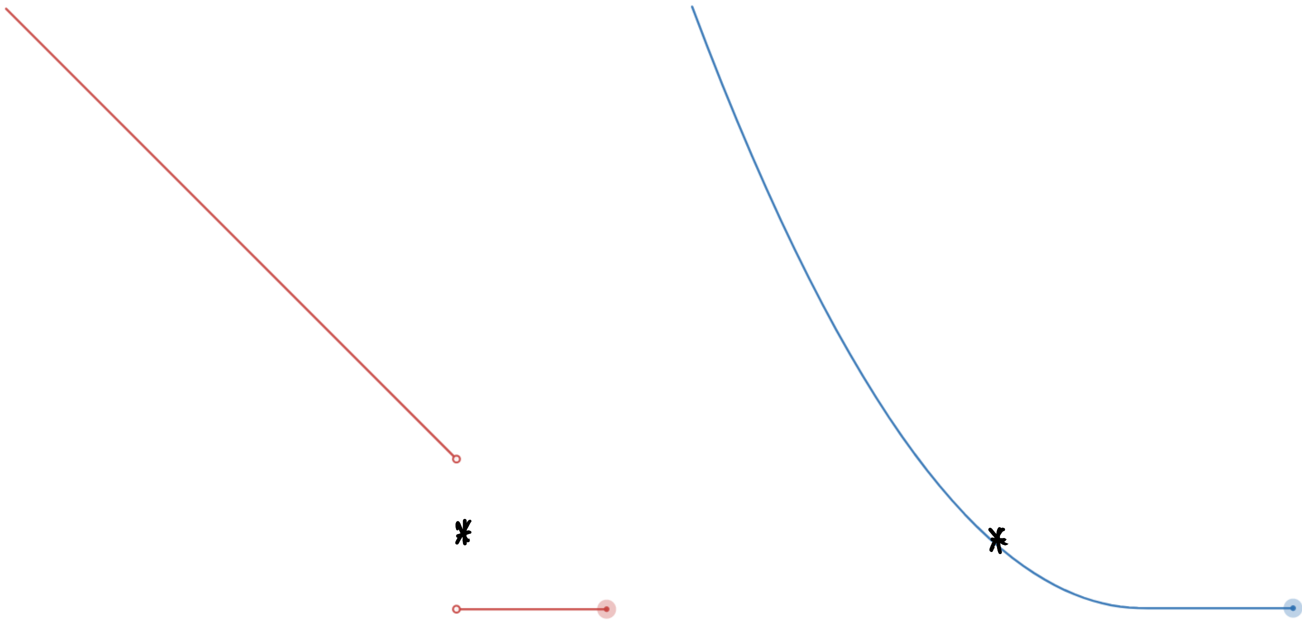
Functions and Limits

1.8 Continuity

Continuity

Example. What are the main difference(s) between the two following curves?

Illustration: https://www.desmos.com/calculator/hflxgbsemz



(a) Graph of f(x). (b) Graph of g(x)

(1) red: break point , blue: no break point

(2) red: not defined at (*) , blue: is defined at *

(3) red: $\lim_{x \to *} f(x) \nexists$, blue: $\lim_{x \to *} f(x) \exists$

(4) is red & the other is blue.

Example. Now, what are the differences between the two following functions?

(a) $f(x) = \begin{cases} 2-x & \text{if } -2 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$

↳ red.

(b) $g(x) = \begin{cases} \frac{4}{9}(1-x)^2 & \text{if } -2 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$

↳ blue.

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

Three things to verify to show a function is continuous:

a) The function is defined at x = a.

Find the domain.

b) The limit of the function exists at x = a.

Use the limit rules.

c) The limit of the function at x = a equals the value of the function at x = a.

Discontinuity:

$x=a$ is a discontinuity of $f(x)$ if

(a) or (b) or (c)

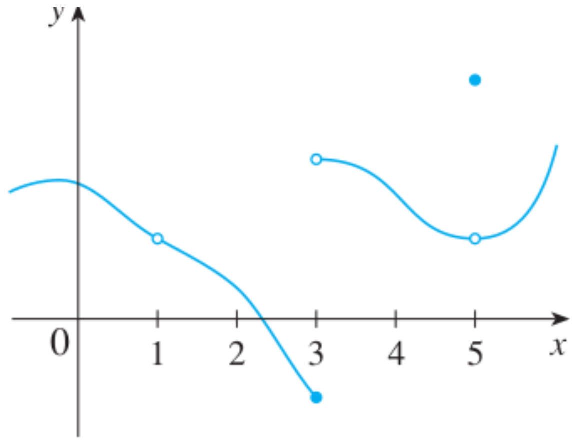
is not satisfied.

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

$\underline{x=1}$ $f(1)$ not defined.

$\underline{x=3}$ $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$.

$\underline{x=5}$ $\lim_{x \to 5} f(x) \neq f(5)$.



Example. Check if the functions in the first example are continuous at x = 1 using the formulas.

(1) (a) f should be defined at $x=1$.

but f is not $\longrightarrow$ discontinuous at $x=1$

(2) (a) g should be defined at $x=1$.

Yes & $g(1)=0$.

(b) $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} \frac{4}{9}(1-x)^2 = 0$

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} 0 = 0$.

$\Rightarrow \lim_{x\to 1} g(x) = 0$.

(c) $g(1) = 0 = \lim_{x\to 1} g(x)$ ✓

So $g(x)$ is continuous at $x=1$.

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

(a) $\text{Dom } f = (-\infty, 2) \cup (2, \infty)$

$f(x) = \dfrac{\cancel{(x-2)}(x+1)}{\cancel{x-2}} = x+1 \quad (x \neq 2).$

Subs. Rule: $\displaystyle\lim_{x \to a} \frac{x^2 - x - 2}{x-2} = \frac{a^2 - a - 2}{a-2} = f(a) \quad (a \neq 2)$

so f is continuous at every point of $(-\infty, 2) \cup (2, \infty)$.

f is discontinuous at $x = 2$ ($f(2) \nexists$).

(b) $\text{Dom } f = (-\infty, \infty)$.

$\displaystyle\lim_{x \to 0} \frac{1}{x^2} = +\infty \neq 1 = f(0) \rightarrow f$ discont. at $x = 0$.

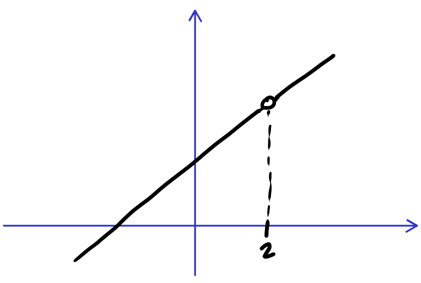
$\boxed{a \neq 0}$

$\displaystyle\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2} = f(a)$ ✓ $\rightarrow f$ cont. at every point of $(-\infty, 0) \cup (0, \infty)$

(c) $\displaystyle\lim_{x \to 0^-} f(x) = 0 \neq 1 = \lim_{x \to 0^+} f(x)$. $\rightarrow$ f is discont. at $x = 0$.

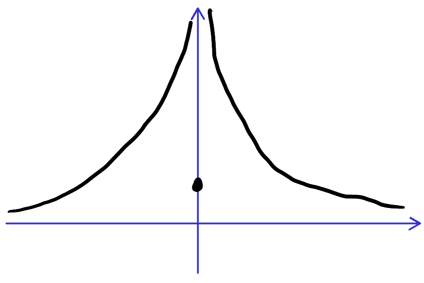
f is cont at all other points in $(-\infty, 0) \cup (0, \infty)$.

3 kinds of discontinuity.



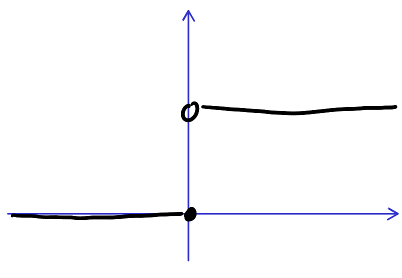
(a) Removable.

$\displaystyle\lim_{x \to 2} f(x) = 3 \ (\exists)$ & $f(2) \nexists$



(b) Infinite discontinuity.

$\displaystyle\lim_{x \to 0} f(x) = +\infty$



(c) Jump discontinuity.

$\displaystyle\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Continuous from left and right.

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

EXAMPLE. Is the function

$$f(x) = \begin{cases} 1 & , \text{ if } x > 0 \\ 0 & , \text{ if } x \leq 0 \end{cases}$$

(a) continuous from the right at x = 0 (b) continuous from the left at x = 0.

(a) $f(0) = 0$, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$

$\Rightarrow$ $0 \neq \lim_{x \to 0^+} f(x)$

$\Rightarrow$ f is not continuous from the right at $x = 0$.

(b) $f(0) = 0$ & $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$

$\Rightarrow$ $f(0) = \lim_{x \to 0^-} f(x)$

$\Rightarrow$ f is continuous from the left at $x = 0$.

Properties of Continuous Functions.

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Consequences:

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

Recall: $\lim_{x \to a} x^n = a^n \longrightarrow f(x) = x^n$ is continuous

Substitution Rule Revisited.

EXAMPLE 5 Find $\lim_{x \to -2} \underbrace{\dfrac{x^3 + 2x^2 - 1}{5 - 3x}}_{f(x)}$.

$\mathrm{Dom}\, f = (-\infty, 5/3) \cup (5/3, \infty)$.

f is rational $\Rightarrow$ f is cont. on dom f.

$\Rightarrow$ f is cont. at $x = -2$.

So,

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = f(-2) = \boxed{-1/11}$$

EXAMPLE 7 Evaluate $\lim_{x \to \pi} \underbrace{\frac{\sin x}{2 + \cos x}}_{f(x)}$.

Dom f.

$2 + \cos x \overset{??}{\neq} 0$ because of (*)

$-1 \leq \cos x \leq 1$

$\Rightarrow \boxed{1 \leq 2 + \cos x \leq 3 \quad (*)}$

So f is continuous on $(-\infty, \infty) = \text{Dom} f$.

$$\Rightarrow \lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = f(\pi) = \frac{\sin(\pi)}{2 + \cos(\pi)} = \frac{0}{2 - 1} = \boxed{0}$$

8 Theorem If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at g(a), then the composition f(g(x)) is continuous at a.

Example. Find the value of

$$\lim_{x\to 1/2} \sin(\pi - \pi x^2)$$

$f(x) = \sin x \longrightarrow$ continuous on $(-\infty, \infty)$

$g(x) = \pi - \pi x^2 \longrightarrow$ continuous on $(-\infty, \infty)$

$\Rightarrow \quad h(x) = \underbrace{f(g(x))}_{\to \sin(\pi - \pi x^2)}$ is continuous on $(-\infty, \infty)$

$\Rightarrow \quad h(x)$ is continuous at $1/2$.

So,

$$\lim_{x\to 1/2} \sin(\pi - \pi x^2) \overset{\text{cont.}}{=} \sin\left(\pi - \pi\left(\frac{1}{2}\right)^2\right)$$

$$= \sin\left(\frac{3\pi}{4}\right) \leftarrow \pi - \frac{\pi}{4}$$

$$= \boxed{\frac{\sqrt{2}}{2}}$$

The Intermediate Value Theorem

EXAMPLE. Suppose we have a function

$$f(x) = x^2 - 1.$$

Does the graph of the function f cross the horizontal line y = 3?

Partial Answer: Yes, using the graph.

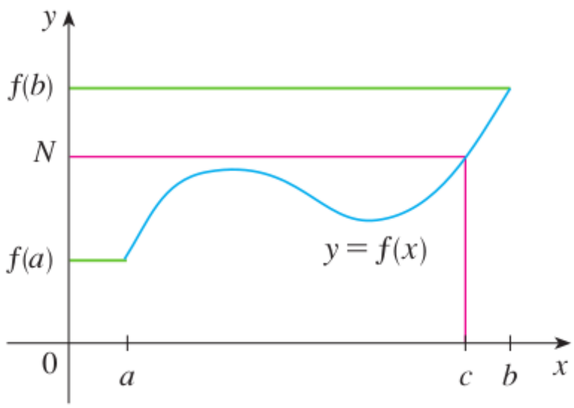
Complete Answer: Yes because

1. $x^2 - 1 \rightarrow$ polynomial $\rightarrow$ continuous on $(-\infty, \infty)$
2. $x_1 = 1 \rightarrow f(1) = 0$
   $x_2 = 4 \rightarrow f(4) = 15$
3. $f(1) < 3 < f(4)$

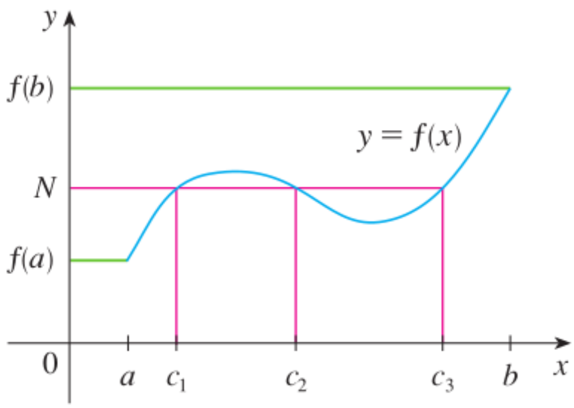
$\hookrightarrow$ so $f(x)$ must cross the line $y = 3$

$\hookrightarrow$ there is a r in dom f & $1 < r < 4$ such that $f(r) = 3 \Rightarrow \boxed{r^2 - 1 = 3}$.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



(a) Find one number c



(b) Find multiple numbers c

EXAMPLE 9 Show that there is a root of the equation

$$\underbrace{4x^3 - 6x^2 + 3x - 2}_{f(x)} = 0$$

between 1 and 2.

root: a number c s.t. $4c^3 - 6c^2 + 3c - 2 = 0$.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$

$N = 0$

$a = 1$

$b = 2$

1. $f(x)$ continuous? YES, a polynomial.
2. $f(1) = -1$, $f(2) = 12$ $\neq$
3. $-1 < 0 < 12$ ✓

So, from the intermediate Value Theorem (IVT), there is a c between 1 and 2 such that

$$f(c) = 0$$

$$\Rightarrow \quad 4c^3 - 6c^2 + 3c - 2 = 0$$

for $1 < c < 2$.