Chapter 2

Derivatives

2.2 The Derivatives as a Function

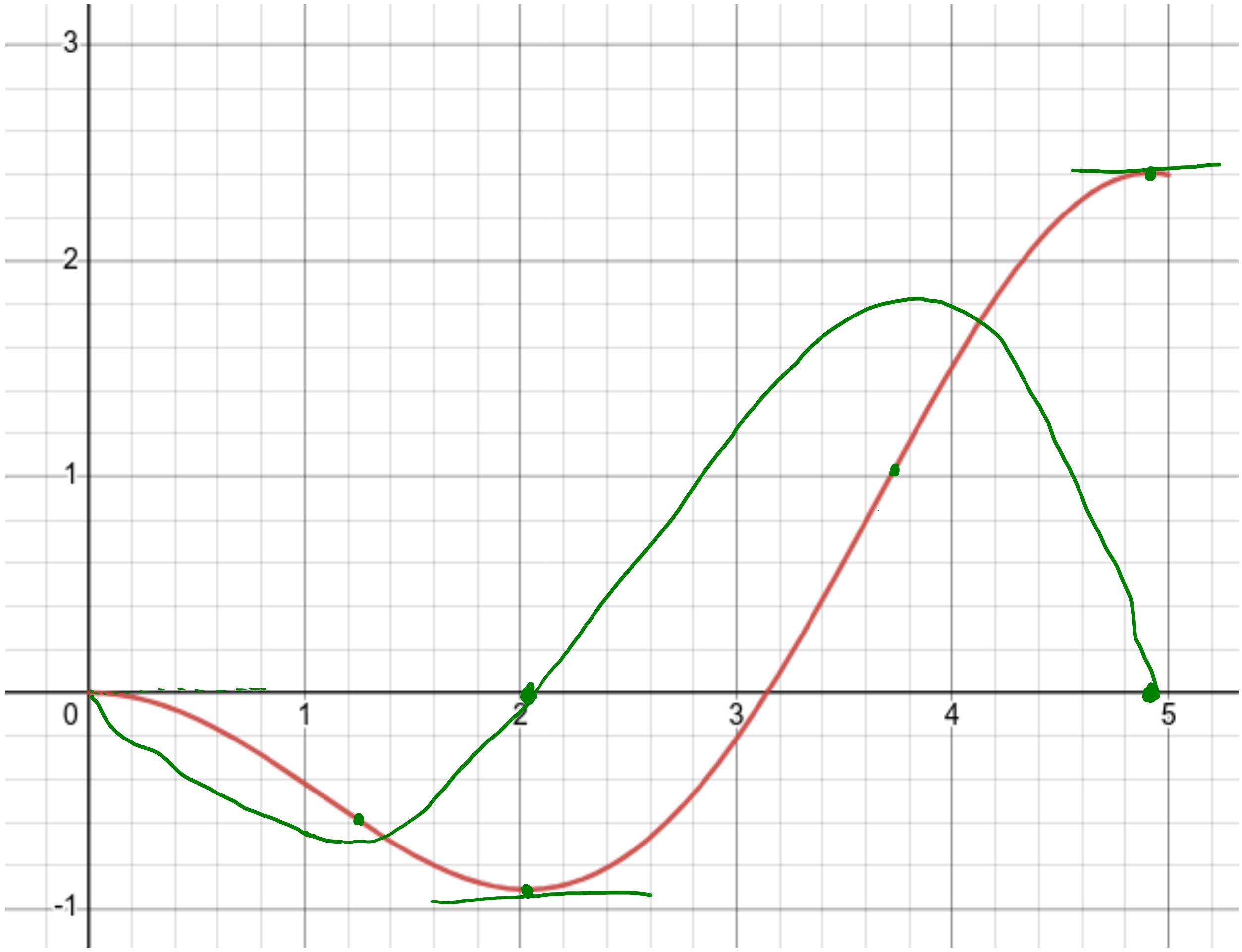
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Dom of f': $\{x : f'(x) \text{ exists}\}$.

EXAMPLE 1 The graph of a function f is given . Use it to sketch the graph of the derivative f'.

Desmos: https://www.desmos.com/calculator/o7lfvk2sar

TRICK: $f'(x)$ is the slope of the tangent line passing through $(x, f(x))$.



EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

(b) Illustrate this formula by comparing the graphs of f and f'. (Do it with Desmos)

(a)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x}+h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} \quad (x \neq 0)$$

$$= \frac{1}{2\sqrt{x}} \Rightarrow \boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

$$\text{Dom } f' = (0, \infty).$$

(b) With Desmos.

Other notations for the derivative.

Leibniz's Notation.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

EXAMPLE. What is the value of $\left.\frac{dy}{dx}\right|_{x=2}$ if $y = x^2$.

$$\left.\frac{dy}{dx}\right|_{x=2} = f'(2).$$

① Formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= 2x$$

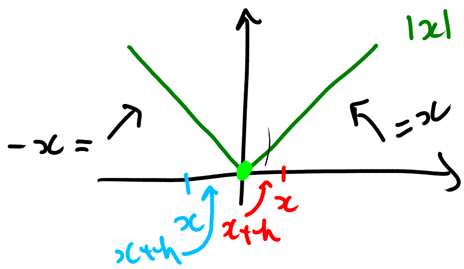
② $\frac{dy}{dx} = 2x \Rightarrow \boxed{\left.\frac{dy}{dx}\right|_{x=2} = 4}$

Leibniz Notation: $$\left.\frac{dy}{dx}\right|_{x=a} = f'(a).$$

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

Graph :



Case 1

$x > 0$. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h}$$

$$= 1$$

$\Rightarrow$ f is differentiable on $(0, \infty)$.

Case 2

$x < 0$. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= -1$$

$\Rightarrow$ f is differentiable on $(-\infty, 0)$.

Case 3

$x = 0$.

Left. $$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{h} = -1$$

$\neq$ $\Rightarrow$ $f'(0)$ doesn't exist

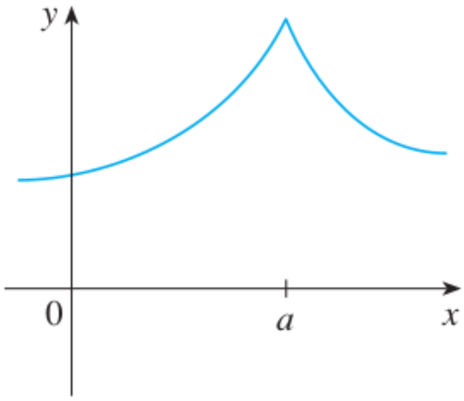
Right $$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1$$

Important Result:

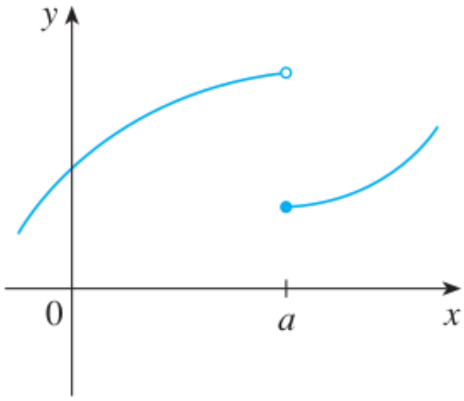
4 Theorem If f is differentiable at a, then f is continuous at a.

Remark: f is continuous at a does not imply that f is differentiable at a.

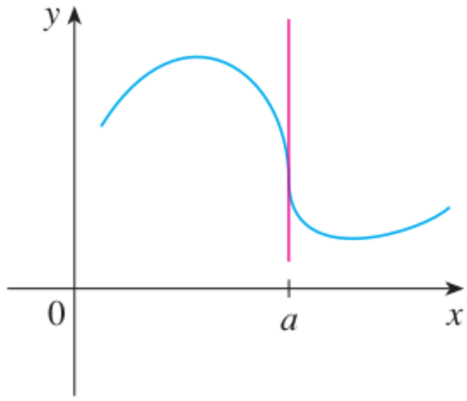
How can a Function Fail to be diffentiable?



(a) A corner



(b) A discontinuity



(c) A vertical tangent

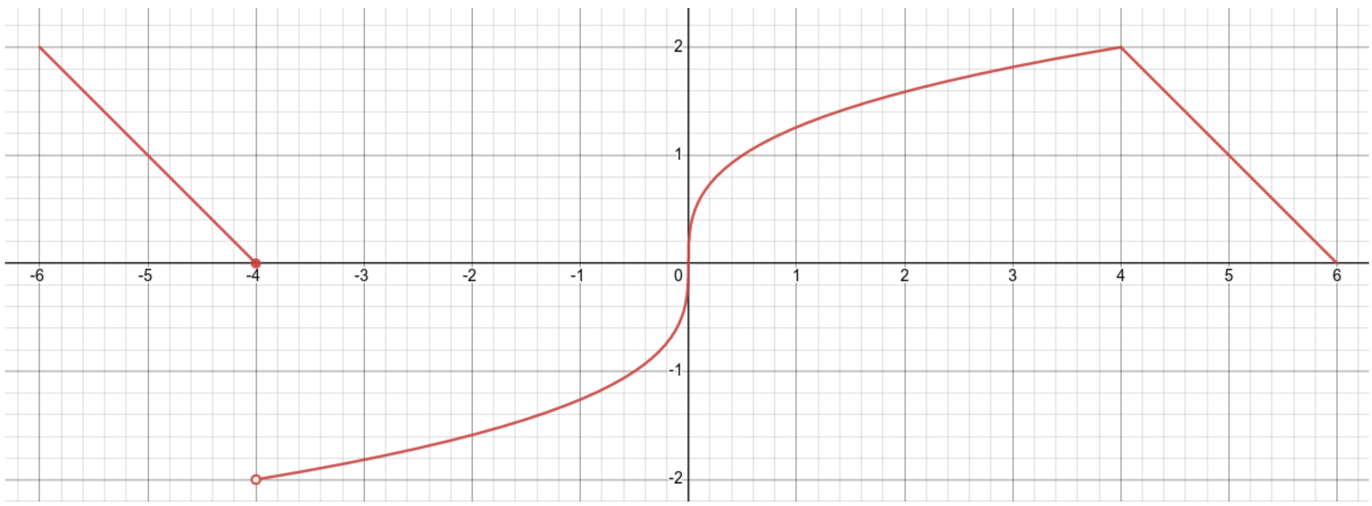
(a) when $\lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h} \neq \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

(b) when f is discontinuous at $x = a$.

(c) when $\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h} = \pm\infty$

or $\lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h} = \pm\infty$

Example. The graph of the function is given. State, with reasons, the numbers at which the function is NOT differentiable.
Desmos: https://www.desmos.com/calculator/d0aztxzxta



① $x = -4$ (Jump discontinuity).

② $x = 0$ (infinite slope or vertical tangent line)

③ $x = 4$ (There is a corner or $\lim_{h \to 0^-} (\ldots) \neq \lim_{h \to 0^+} (\ldots)$.

Higher Derivatives.

Second derivative:

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}}\underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

Other notations:

$y^{(2)}$ or y''

$f'(x) \to f''(x)$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

f' $\quad f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \cdots = 3x^2 - 1$

f'' $\quad f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$

$= \lim_{h\to 0} \frac{3(x+h)^2 - 1 - (3x^2 - 1)}{h}$

$= \cdots = 6x$

Acceleration: Rate change of velocity

$s(t)$: position.

$v(t)$: velocity.

$a(t)$: acceleration

$\Rightarrow \quad a(t) = \underbrace{v'(t)}_{\frac{dv}{dt}} = \underbrace{s''(t)}_{\frac{d^2s}{dt^2}}.$

Third Derivative.

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Jerk: $j = \frac{da}{dt} = \frac{d^3s}{dt^3}$

n-th Derivative.

$$y^{(n)} = f^{(n)}(x) = \boxed{\frac{d^ny}{dx^n}}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

We know that $f'(x) = 3x^2 - 1$

$f''(x) = 6x$

$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h}$$

$$= \lim_{h \to 0} \frac{6(x+h) - 6x}{h} = 6$$

$$f''''(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h}$$

$$= \lim_{h \to 0} \frac{6 - 6}{h} = 0$$