Chapter 2

Derivatives

2.3 Differentiation Formulas

Constant Function.

$$f(x) = c \quad \Rightarrow \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

$$= 0$$

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions. $f(x) = x^n$

n = 1. $f(x) = x$

$$\lim_{h \to 0} \frac{x + h - x}{h} = 1 \quad \Rightarrow \quad \frac{d}{dx}(x) = 1$$

n = 2. $f(x) = x^2$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2x\cancel{h} + \cancel{h^2}^{\,h}}{\cancel{h}} = 2x$$

$$\Rightarrow \quad \frac{d}{dx}(x^2) = 2x$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE. Find the derivative of $y = x^4 - 6x^2 + 4. = f(x)$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^4 - 6(x+h)^2 + 4 - (x^4 - 6x^2 + 4)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} - 6\,\frac{(x+h)^2 - x^2}{h} + \frac{4-4}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} - 6\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} + \lim_{h\to 0} \frac{4-4}{h}$$

$$= \frac{d}{dx}(x^4) - 6\,\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$$

$$= 4x^3 - 6\cdot 2x + 0$$

$$= 4x^3 - 12x$$

$$= \boxed{4x(x^2 - 3)}$$

Multiplication by a constant.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Sum.

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Difference.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Product.

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Caution!!!

$$\cancel{\frac{d}{dx}(fg) \neq \frac{d}{dx}(f)\frac{d}{dx}(g).}$$

Example.

$f(x) = x \qquad g(x) = x$

$f(x)g(x) = x^2$

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df}{dx}\,\frac{dg}{dx} = 1\cdot 1 = 1$$

$$\frac{d}{dx}(x^2) = 2x$$

not the same

Example. Find the derivative of the function $h(x) = \underbrace{(5x^2 - 2)}_{f(x)}\underbrace{(x^3 + 3x)}_{g(x)}$.

$$\frac{dh}{dx} = f(x)\frac{d g(x)}{dx} + g(x)\frac{d}{dx}f(x)$$

$$= (5x^2-2)\frac{d}{dx}(x^3+3x) + (x^3+3x)\frac{d}{dx}(5x^2-2)$$

$$= (5x^2-2)(3x^2+3) + (x^3+3x)(10x)$$

$$= 15x^4 + 15x^2 - 6x^2 - 6 + 10x^4 + 30x^2$$

$$= \boxed{25x^4 + 39x^2 - 6}$$

Quotient.

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Caution !!

$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{\frac{d}{dx}(f)}{\frac{d}{dx}(g)}$$

Example.

$f(x) = x^3$ & $g(x) = x$

$f' = 3x^2$, $g' = 1$

$$\frac{d}{dx}\left(\frac{x^3}{x}\right) = \frac{d}{dx}(x^2) = 2x \neq \frac{3x^2}{1}$$

EXAMPLE 8 Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Compute the derivative.

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2+x-2)\,(x^3+6) - (x^2+x-2)\,\frac{d}{dx}(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

General Power rule.

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Case n = 0:

$$\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0.$$

Example. Find the derivative of the function $f(x) = x^{2/3}$.

$n = 2/3$

$$\Rightarrow \quad f'(x) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$$

$$\Rightarrow \quad f'(x) = \frac{2}{3x^{1/3}} = \boxed{\frac{2}{3\sqrt[3]{x}}}$$

$$\frac{1}{x^{2/3}} = \frac{1}{(x^2)^{1/3}} = \frac{1}{\sqrt[3]{x^2}}$$

EXAMPLE 13 At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

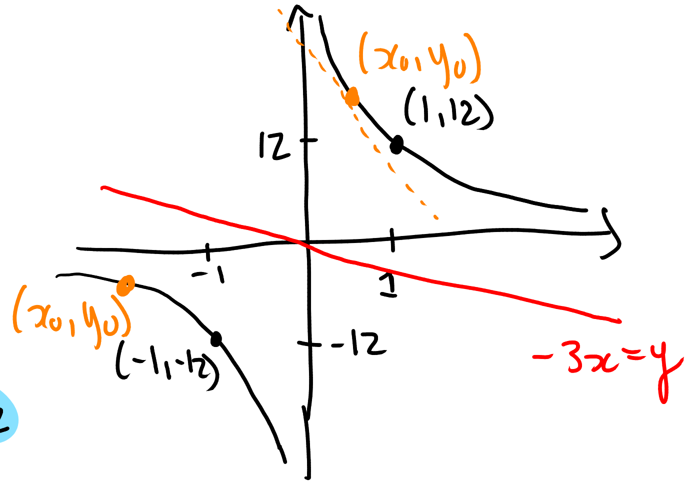
goal: find (x_0, y_0) on the hyperbola.

① Graph.

$$y = \frac{12}{x} \quad (x \neq 0)$$

$$y = -3x$$

$$y' = \frac{d}{dx}\left(12x^{-1}\right) = -12x^{-2}$$



② How are two line parallel?

$$y_1 = m_1 x + b_1 \quad \& \quad y_2 = m_2 x + b_2$$

$$y_1 \parallel y_2 \quad \text{if} \quad m_1 = m_2 .$$

③ Find the points.

$y_1 = -3x$

$y_2 = f'(x_0)\, x + b$

So, $-3 = f'(x_0) = \dfrac{-12}{x_0^2}$

$\Rightarrow x_0^2 = 4$

$\Rightarrow x_0 = \pm 2$

Summary of Differentiation Formulas.

Table of Differentiation Formulas

$\dfrac{d}{dx}(c) = 0$	$\dfrac{d}{dx}(x^n) = nx^{n-1}$	
$(cf)' = cf'$	$(f+g)' = f' + g'$	$(f-g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$	

$$x_0 = 2 \Rightarrow \quad y_0 = \frac{12}{2} = 6 \longrightarrow \boxed{(2,6)}$$

$$x_0 = -2 \Rightarrow \quad y_0 = \frac{12}{-2} = -6 \longrightarrow \boxed{(-2,-6)}$$