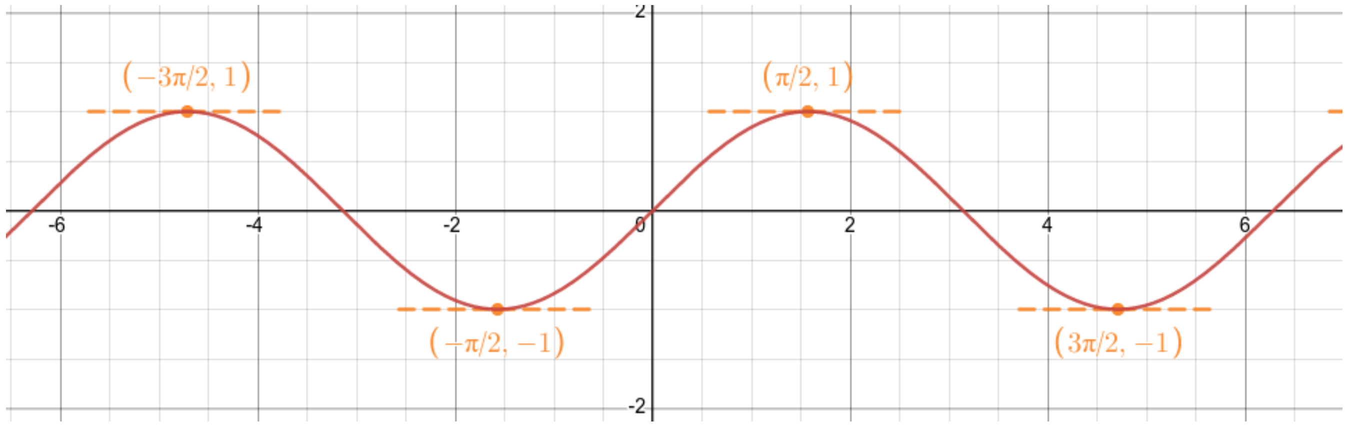
Chapter 2
Derivatives
2.4 Derivatives of Trigonometric Functions

Derivative of the Sine function.



Desmos: https://www.desmos.com/calculator/mhbl7c2hzy

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Proof.

By def.

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Trig ident.:

$$\sin(\underset{\substack{\uparrow \\ A}}{x} + \underset{\substack{\uparrow \\ B}}{h}) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

$$\Rightarrow \quad \sin(x+h) - \sin x = \sin x \cos h + \sin h \cos x - \sin x$$

$$= \sin x(\cos h - 1) + \cos x \sin h$$

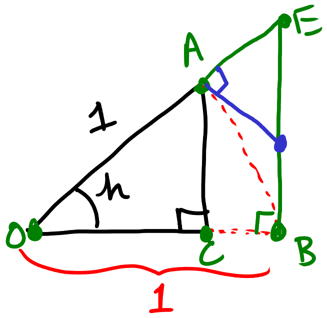
So,

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h}$$

$$= \sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{2} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1}$$

① We can show that

$$\cos h \le \frac{\sin h}{h} \le 1 .$$



Now $\displaystyle\lim_{h\to 0} \cos h = \cos(0) = 1$

and $\displaystyle\lim_{h\to 0} 1 = 1$

By Squeeze Thm., $\displaystyle\lim_{h\to 0} \frac{\sin h}{h} = 1 .$

② $\displaystyle \cos h - 1 = -2\left(\frac{1-\cos h}{2}\right) = -2\sin^2(h/2)$ (trig identity)

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2(h/2)}{h}$$

$$= -\lim_{h\to 0} \frac{\sin(h/2)\,\sin(h/2)}{h/2}$$

$$= -\lim_{h\to 0} \frac{\sin(h/2)}{h/2} \lim_{h\to 0} \sin(h/2)$$

$$= -\lim_{u\to 0} \frac{\sin(u)}{u} \lim_{h\to 0} \sin(h/2)$$

$$= -(1)\cdot \sin(0) = 0$$

Conclusion.

$$\frac{d}{dx}(\sin x) = \sin x \cancelto{=0}{\lim_{h\to 0} \frac{\cos h - 1}{h}} + \cos x \cancelto{=1}{\lim_{h\to 0} \frac{\sin h}{h}}$$

$$= \cos x .$$

□

Trigonometric Functions (reminder).

- $\sec x = \dfrac{1}{\cos x}$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\csc x = \dfrac{1}{\sin x}$
- $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

Derivatives of Other Trigonometric Functions.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Proof for the formula for f(x) = tan(x).

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

trig. identity.

$$= \frac{1}{\cos^2 x} = \sec^2 x .$$

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

(a) $$f'(x) = \frac{\frac{d}{dx}(\sec x)(1+\tan x) - \sec x \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x\,(1+\tan x) - \sec x(\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x(\tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{\sec x \tan x + \sec x\left(\frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}\right)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x\left(\frac{-\cos^2 x}{\cos^2 x}\right)}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x - \sec x}{(1+\tan x)^2} = \frac{\tan x - 1}{\cos x(\tan x + 1)^2}$$

(2) tangent is horizontal if $f'(x) = 0$

when $\tan x - 1 = 0 \iff \tan x = 1$

$\iff \sin x = \cos x$

$\iff \boxed{x = \frac{\pi}{4} + n\pi}$

EXAMPLE 6 Calculate $\lim_{x \to 0} x \cot x$.

$$x = \frac{1}{(1/x)} = \frac{1}{x^{-1}}$$

(1) $$\lim_{x \to 0} x \cot x = \lim_{x \to 0} x \frac{\cos x}{\sin x}$$

$$= \lim_{x \to 0} \frac{\cos x}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\lim_{x \to 0} \cos x}{\lim_{x \to 0} \frac{\sin x}{x}} = \frac{1}{1} = \boxed{1}$$

(2) See the limit as the derivative of some function.

$$x \cot(x) = \frac{x}{\tan x} \xrightarrow{\text{flipped}} \frac{\tan x}{x}$$

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\tan x - \tan 0}{x - 0}$$

$$= \frac{d}{dx}(\tan x) \Big|_{x=0}$$

$$= \sec^2(x) \Big|_{x=0}$$

$$= \frac{1}{1^2} = \boxed{1}$$

$$\Rightarrow \lim_{x \to 0} \frac{x}{\tan x} = \frac{1}{1} = 1$$