Chapter 2
Derivatives
2.5 Chain Rule

How do you differentiate the function $F(x) = \sqrt{x^2+1}$?

$$F'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{(x+h)^2+1} - \sqrt{x^2+1}\right)\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{(x+h)^2+1 - (x^2+1)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \boxed{\frac{1 \cdot 2x}{2\sqrt{x^2+1}}}$$

__Another point of view:__

$$F(x) = g(f(x)) \qquad = 2x = f'(x)$$

$$f(x) = x^2+1 \qquad = \frac{1}{2\sqrt{x^2+1}} = g'(x^2+1)$$

$$g(x) = \sqrt{x} \rightarrow g'(x) = \frac{1}{2\sqrt{x}}$$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Main idea:

$$\frac{d}{dx}\underbrace{f}_{\text{outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

(a) $$\frac{dy}{dx} = \frac{d}{du}\sin(u)\Big|_{u=x^2} \cdot \frac{du}{dx}$$

$$= \cos(x^2) \cdot 2x$$

(b) $$y = (\sin x)^2 \longrightarrow f(x)$$

$\sin x \downarrow g(x)$

$f(x) = x^2 \longrightarrow f'(x) = 2x$

$g(x) = \sin x \longrightarrow g'(x) = \cos x$

$$\frac{dy}{dx} = f'(g(x))\ g'(x)$$

$$= 2(\sin x)\cos x$$

EXAMPLE 4 Find $h'(x)$ if $h(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$. $= \dfrac{1}{(x^2+x+1)^{1/3}}$

$$= (x^2+x+1)^{-1/3}$$

$f(x) = x^{-1/3} \longrightarrow f'(x) = \frac{-1}{3}x^{-4/3}$

$g(x) = x^2+x+1 \longrightarrow g'(x) = 2x+1$

$$\frac{dh}{dx} = f'(g(x)) \cdot g'(x)$$

$$= \frac{-1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$$

EXAMPLE 6 Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$.

$(2x+1)^5 \longrightarrow$ Ex~~pa~~nd

$$\frac{dy}{dx} = \left[\frac{d}{dx}(2x+1)^5\right]\left[x^3-x+1\right]^4 + (2x+1)^5 \frac{d}{dx}(x^3-x+1)^4$$

$$= 5(2x+1)^4 \cdot \frac{d}{dx}(2x+1) \left[x^3-x+1\right]^4$$

$$+ (2x+1)^5 \, 4\,(x^3-x+1)^3 \frac{d}{dx}(x^3-x+1)$$

$$= 5(2x+1)^4 \cdot (2) \, (x^3-x+1)^4$$

$$+ (2x+1)^5 \cdot 4 \cdot (x^3-x+1)^3 \, (3x^2-1)$$

$$= 10(2x+1)^4(x^3-x+1)^4 + (2x+1)^5(x^3-x+1)^3 \cdot (12x^2-4).$$