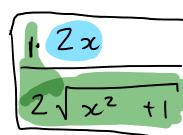
Chapter 2 Derivatives

2.5 Chain Rule

How do you differentiate the function $F(x) = \sqrt{x^2 + 1}$? $F'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2 + 1} - \sqrt{x^2 + 1}}{h}$ $= \lim_{h \to 0} \frac{(\sqrt{(x+h)^2 + 1} - \sqrt{x^2 + 1})(\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1})}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}}$ $= \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}}$ $= \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}}$ $= \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}}$ $= \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}}$ $= \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}}$ $= \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{\sqrt{(x+h)^2 + 1} + \sqrt{x^2 + 1}}$

$$= \lim_{N \to 0} \frac{2xh + h^2}{h((x+h)^2 + 1 + \sqrt{x^2 + 1})} = \frac{1}{h((x+h)^2 + 1 + \sqrt{x^2 + 1})}$$



Another point of new:

Another point of view.

$$F(x) = g(f(x))$$

$$= 2x = f'(x)$$

$$f(x) = x^{2}+1$$

$$= \frac{1}{2\sqrt{x^{2}+1}} = g'(x^{2}+1)$$

$$g(x) = \sqrt{x} - g'(x) = \frac{1}{2\sqrt{x}}$$

The Chain Rule If g is differentiable at x and f is differentiable at g(x), then the composite function $F = f \circ g$ defined by F(x) = f(g(x)) is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if y = f(u) and u = g(x) are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Main idea:

$$\frac{d}{dx} \quad f \qquad (g(x)) \qquad = \qquad f' \qquad (g(x)) \qquad \cdot \qquad g'(x)$$
outer function
evaluated at inner function
of outer of outer function
derivative of outer at inner function

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

(a)
$$\frac{dy}{dx} = \frac{d}{du} \sin(u) \Big|_{u=x^2} \frac{du}{dx}$$

$$= \cos(x^2) \cdot 2\pi$$
(b) $y = (\sin x)^2 \qquad f(x)$

$$= f(x) = x^2 - x \qquad f'(x) = 2x$$

$$g(x) \qquad g(x) = \sin x - x \qquad g'(x) = \cos x$$

$$= 2(\sin x) \cos x$$

EXAMPLE 4 Find
$$h'(x)$$
 if $h(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}} = \frac{1}{(x^2 + x + 1)^{1/3}}$

$$= (x^2 + x + 1)$$

$$f(x) = x^{-1/3} - f'(x) = \frac{1}{3} x^{-1/3}$$

$$g(x) = x^2 + x + 1 - y + y + y = 2x + 1$$

$$\frac{dh}{dx} = f'(g(x)) \cdot g'(x)$$

$$= -\frac{1}{3} (x^2 + x + 1)^{-1/3} (2x + 1)$$

EXAMPLE 6 Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$.

$$\frac{dy}{dx} = \left[\frac{d}{dx}(2x+1)^{5}\right]\left[x^{3}-x+1\right]^{4} + (2x+1)^{5}\frac{d}{dx}(x^{2}x+1)^{4}$$

$$= 5(2x+1)^{4} \cdot \frac{d}{dx}(2x+1) \cdot \left[x^{3}-x+1\right]^{4}$$

$$+ (2x+1)^{5} \cdot 4 \cdot \left[x^{3}-x+1\right]^{3}\frac{d}{dx}(x^{3}-x+1)$$

$$= 5(2x+1)^{4} \cdot (2) \cdot \left(x^{3}-x+1\right)^{4}$$

$$+ (2x+1)^{5} \cdot 4 \cdot \left(x^{3}-x+1\right)^{4}$$

$$+ (2x+1)^{5} \cdot 4 \cdot \left(x^{3}-x+1\right)^{4} \cdot (2x+1)^{5} \cdot (3x^{2}-1)$$

$$= 10(2x+1)^{4} \cdot \left(x^{3}-x+1\right)^{4} + (2x+1)^{5} \cdot \left(x^{3}-x+1\right)^{3} \cdot (12x^{2}-4)$$