Math 241

Chapter 2

Section 2.7: Rates of Change

Contents

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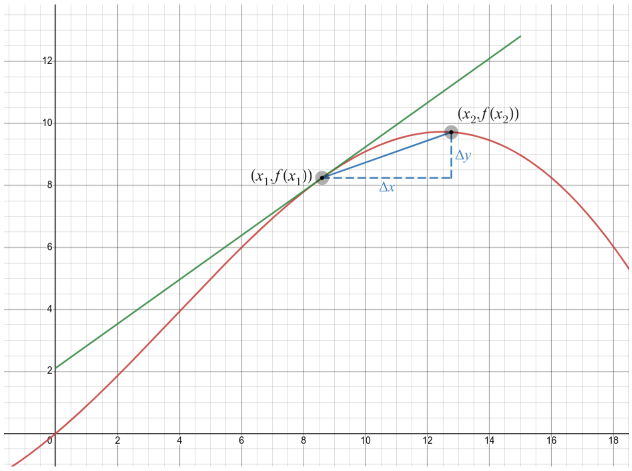
Rate of Change and Derivative

Let $y = f(x)$ be a function.

- If x goes from x_1 to x_2, then the change in x is
$$\Delta x = x_2 - x_1.$$
- When x changes from x_1 to x_2, then y changes from $f(x_1)$ to $f(x_2)$ and the change in y is
$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1).$$
- The **average rate of change** at x_1 is therefore $\dfrac{\Delta y}{\Delta x}$.
- The **instantaneous rate of change** at x_1 is
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$



https://www.desmos.com/calculator/ajsf8ggdwy

Remark: The name of the variables may be different. We can use the variables x, t (or other letters) for the independent variable and y, s (or other letters) for the dependent variable.

Example 1. The position s of an object is given by the function $s = f(t) = t^2$

a) Compute the average rate of change at $t_1 = 1$ if $t_1 = 1$ and $t_2 = 2$.

b) Compute the instantaneous rate of change at $t_1 = 1$.

a) $$\frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{4-1}{2-1} = \frac{3}{1} = 3 \text{ u/s}$$

b) $$\left.\frac{ds}{dt}\right|_{t=1}$$

$$\frac{ds}{dt} = \frac{d}{dt}(t^2) = 2t$$

$$\Rightarrow \left.\frac{ds}{dt}\right|_{t=1} = 2 \text{ m/s}$$

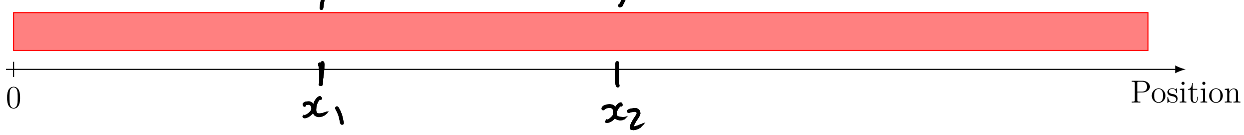
Remarks: Let the position s of an object be given by $s = f(t)$ where f is a function of time t.

- The average velocity at t_1 is the average rate of change in s.
- The instantaneous velocity at t_1 is the instantaneous rate of change at t_1.

Linear Density

Consider a rod.

$m_1 = f(x_1)$ $\qquad$ $m_2 = f(x_2)$



- The position on the rod from the extremity 0 is given by x.
- The mass of the part of the rod from 0 to x is given by

$$m = f(x).$$

Question: How is the mass distributed along the rod?

- The **average linear density** between x_1 and x_2 is the average rate of change in the mass between x_1 and x_2. $\rightarrow \frac{\Delta m}{\Delta x}$
- The **linear density** at x_1 is the instantaneous rate of change in the mass at x_1. $\rightarrow \left.\frac{dm}{dx}\right|_{x=x_1}$

Example 2. A rod as in the figure above has a mass m ($\rightarrow$ kg) given by $f(x) = x^3$.

a) Find the average linear density between $x_1 = 1$ and $x_2 = 2$. $\rightarrow$ cm.

b) Find the linear density at $x_1 = 1$.

a) $$\frac{\Delta m}{\Delta x} = \frac{m_2 - m_1}{x_2 - x_1} = \frac{2^3 - 1^3}{2 - 1} = \frac{7}{1} = 7 \text{ kg/cm}$$

b) $$\left.\frac{dm}{dx}\right|_{x=1}, \qquad \frac{dm}{dx} = \frac{d}{dx}(x^3) = 3x^2$$

$$\Rightarrow \left.\frac{dm}{dx}\right|_{x=1} = 3(1)^2 = 3 \text{ kg/cm}$$

Rate of Change and Epidemics

Suppose there is a virus spreading in a population. We are interested in describing:

- The rate at which the virus spreads from one individual to another at a specific moment in time
- If we know the quantity of infected individuals, denoted by $Q(t)$.

In this case, the rate at which the virus spreads between two days, day t_1 and t_2 respectively, is given by the average rate of change in Q:

$$\frac{\Delta Q}{\Delta t}.$$

Therefore, the rate at which the virus spreads at day t_1 is given by the instantaneous rate of change (when $\Delta t \to 0$):

$$\frac{dQ}{dt}\ !$$

Example 3. Suppose a virus is spreading in the population of deers on Moloka'i. Suppose the number of infected deer at day t is given by $Q(t) = (50/\pi)\sin(\pi t) + 60$. Let $t = 0$ be the first day we observed the presence of the virus and the model is valid up to $t = 5$.

a) Find at which rate the virus spreads in the population at day $t = 3$.

b) Estimates the number of deers infected at day $t = 6$.

a) $\left.\frac{dQ}{dt}\right|_{t=3}$.

$$\frac{dQ}{dt} = \frac{d}{dt}\left(\frac{50}{\pi}\sin(\pi t) + 60\right)$$

$$= \frac{50}{\pi}\cos(\pi t)\cdot\frac{d}{dt}(\pi t)$$

$$= 50\cos(\pi t).$$

$\hookrightarrow$ $50\cos(3\pi)$

$= \boxed{-50 \text{ deer/day}}$

b) $\left.\frac{dQ}{dt}\right|_{t=5} = -50$ and $Q(5) = 60$

$\Rightarrow Q(6) \approx 60 - 50(6-5) = 60 - 50 = \boxed{10 \text{ deers}}$