Chapter 2

Derivatives

2.8 Related Rates.

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

① Identify what is given and is unknown

- Vol ↑ at a rate of 100 cm/s
- rate of change of the radius ??

V : volume of the balloon (cm^3)
r : radius of the balloon (cm)

- $\frac{dV}{dt} = 100 \ cm^3/s$
- $\left.\frac{dr}{dt}\right|_{r=50cm} = ??$

② Connect the variables.

$V = \frac{4}{3}\pi r^3$ (Volume of sphere).

③ Apply $\frac{d}{dt}$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \frac{dV}{dt} = \frac{4\pi}{3}\frac{d}{dt}(r^3) = \frac{4\pi}{3} 3r^2 \frac{dr}{dt}$$

Key Steps.

1) Introduce notation and draw a diagram if possible.

2) Restate the given information and the unknown with the new notation.

3) Connect the variables together with an equation.

4) Apply the chain rule to find the related rates.

5) Plug in the information from step 2.

$$\Rightarrow \quad \frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} \cdot r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad 100 = 4\pi (50)^2 \frac{dr}{dt} \quad \left(\overset{\text{replaced}}{\frac{dV}{dt} = 100 \ \& \ r = 50} \right)$$

$$\Rightarrow \quad \left.\frac{dr}{dt}\right|_{r=50} = \frac{25}{\pi (50)^2} = \frac{\cancel{25}}{\pi \cdot \cancel{5}^2 \cdot 10^2}$$

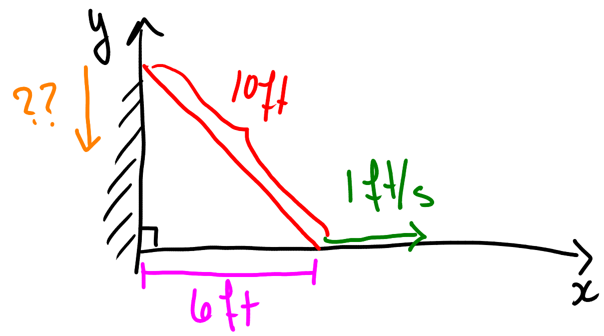
$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{100\pi} \text{ cm/s}$$

(4) Answer:

$$\boxed{\left.\frac{dr}{dt}\right|_{r=50} = \frac{1}{100\pi} \text{ cm/s}}$$

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

① Diagram & variables

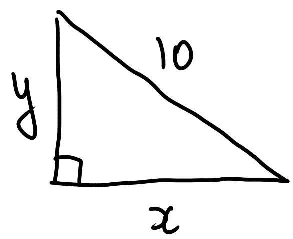


x: distance of the bottom of the ladder to the wall (ft)

y: distance from the top of the latter to the floor (ft)

- Goal: $\left.\frac{dy}{dt}\right|_{x=6} = ???$

② Link.



Pyth. : $x^2+y^2 = 10^2$

$\Rightarrow x^2+y^2 = 100$.

③ Derivative

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(100)$$

$$\Rightarrow \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow 2\cdot 6\cdot 1 + 2y\frac{dy}{dt} = 0 \qquad \left(\begin{array}{c}\frac{dx}{dt}=1 \\ x=6\end{array}\right)$$

$$x^2 + y^2 = 100 \quad \Rightarrow \quad 36 + y^2 = 100$$

$$\Rightarrow \quad y = \sqrt{100 - 36} = \sqrt{64}$$

$$\Rightarrow \quad y = 8$$

So,

$$12 + 2 \cdot 8 \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 16 \frac{dy}{dt} = -12$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{3}{4}$$

(4) Answer

$$\boxed{\frac{dy}{dt} = -\frac{3}{4} \, ft/s}$$

Problem Solving Strategy from the book.

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 3).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.