Chapter 2

Derivatives

2.9 Linear Approximations and Differentials.

An observation:

A curve $y = f(x)$ lies very close to its tangent line near the point of tangency. Linearization https://www.desmos.com/calculator/1sp51krlae

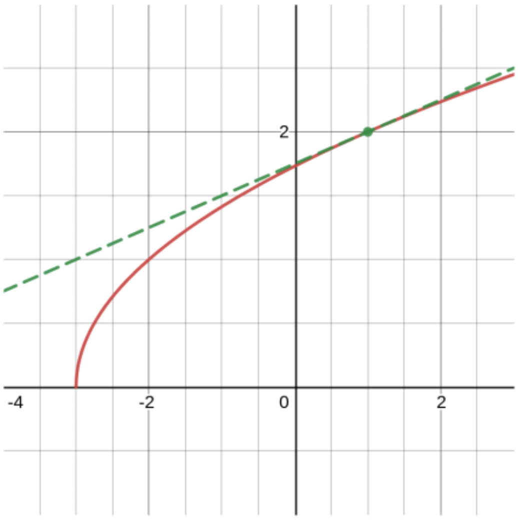


Figure: Linearization near the point of tangency

This suggests to approximate the values of f by the tangent line. This is a really useful procedure because $f(x)$ may be difficult to compute!

$(x_0, y_0) = (a, f(a)) \rightarrow y - f(a) = f'(a)(x-a)$

$\rightarrow y = f(a) + f'(a)(x-a)$

Approximation by the tangent line:

$$f(x) \approx f(a) + f'(a)(x-a)$$

So the linearization is

$$L(x) = f(a) + f'(a)(x-a)$$

$\rightarrow f(x) \approx L(x)$

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

(a) $$f'(x) = \frac{1}{2\sqrt{x+3}} \Rightarrow f'(1) = \frac{1}{4}$$

We have $$f(1) = \sqrt{4} = 2 \rightarrow L(x) = \underset{f(a)}{2} + \frac{1}{4}(x - \underset{a}{1})$$

(b) $$\begin{aligned} \sqrt{3.98} = \sqrt{0.98+3} &= f(0.98) \\ &\approx L(0.98) \\ &= 2 + \frac{1}{4}(0.98-1) \\ &= 2 - \frac{0.02}{4} \\ &= 2 - 0.005 = 1.995 \end{aligned}$$

(c) $$\begin{aligned} \sqrt{4.05} = \sqrt{\underbrace{1.05}_{x} + 3} &= f(1.05) \\ &\approx L(1.05) \\ &= 2 + \frac{1}{4}(\underbrace{1.05}_{x} - 1) \\ &= 2 + \frac{0.05}{4} \\ &= 2 + 0.0125 \\ &= \boxed{2.0125} \end{aligned}$$

<u>CDN</u>

$$\begin{array}{l} 50 \,\lfloor 4 \\ -48 \quad 12.5 \\ 20 \\ -20 \\ \hline 0 \end{array}$$

<u>USA</u>

$$\begin{array}{r} 12.5 \\ 4 \overline{)50.0} \\ -48 \\ \hline 20 \\ -20 \\ \hline 0 \end{array}$$

Differentials.

If $y = f(x)$, then

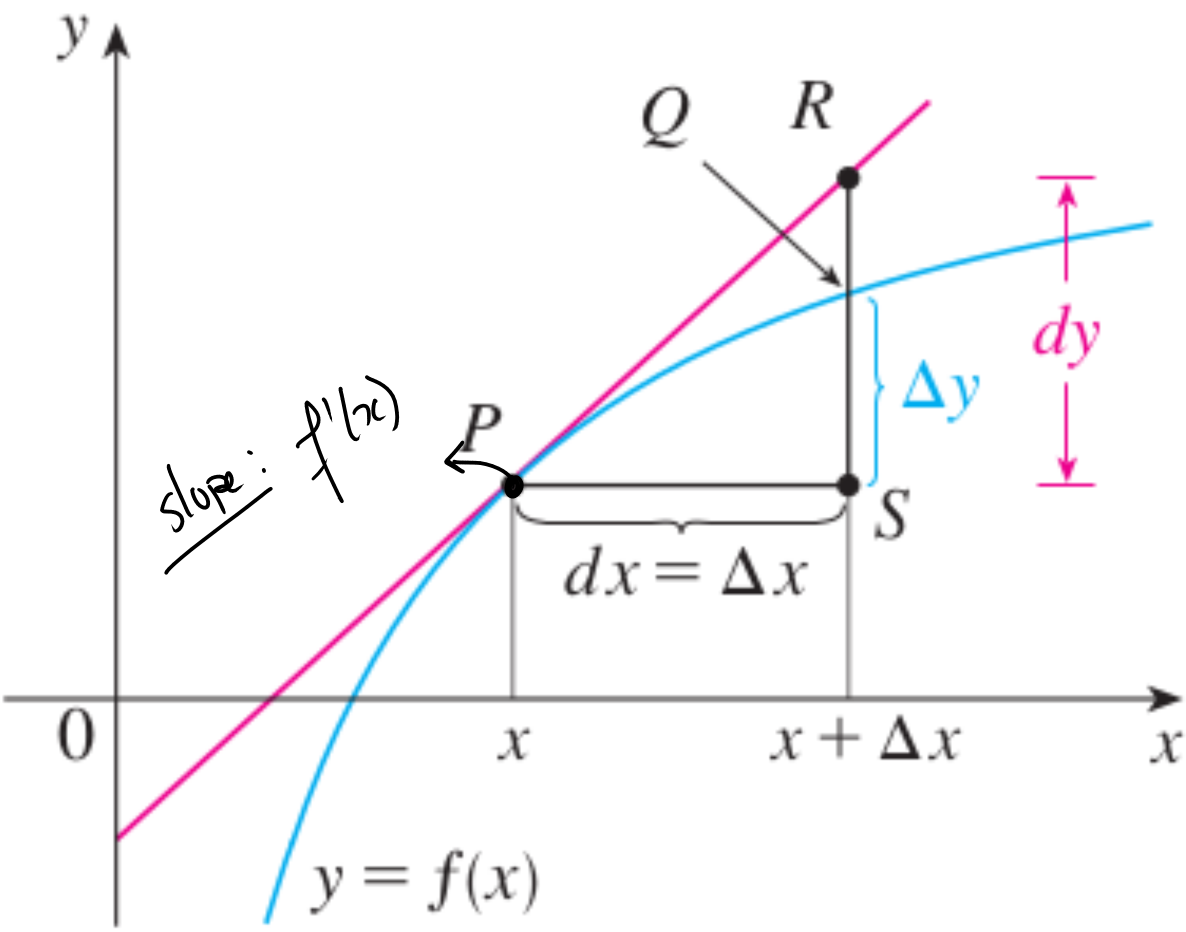
- dx is the differential of x. It's a little increment in the variable x.
- dy is the differential of y and dy is the approximate increment in the variable y given by

$$\frac{dy}{dx} = f'(x) \longrightarrow \boxed{dy = f'(x)dx.}$$

Remark:

$$\Delta y \approx f'(x)\,dx = dy \qquad dx = \Delta x$$

Geometric interpretation.



EXAMPLE 3 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

(a) $f'(x) = 3x^2 + 2x - 2$

$dy = f'(x)\,dx = (3x^2 + 2x - 2)\,dx$

$x = 2$

$dx = \Delta x = 2.05 - 2 = 0.05$

1. $dy = f'(2) \cdot 0.05$

$= (3(4) + 6 - 2) \cdot 0.05 = 0.7$

2. $\Delta y = f(2.05) - f(2)$

$= 0.717675$