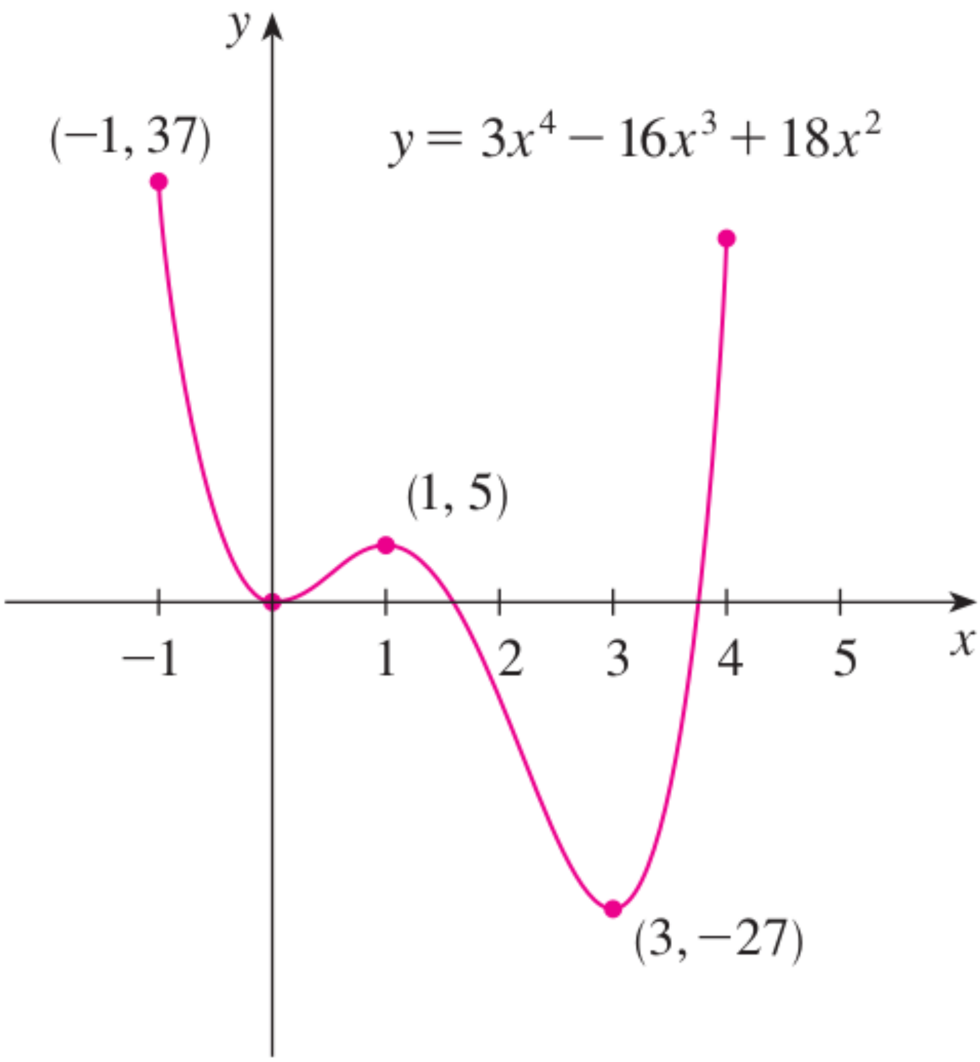
Chapter 3
Applications of Derivatives

3.1 Maximum and Minimum Values

Maximums and minimums.

What would be a maximum value or a minimum value of a function?



Suggestions/observations:

1) At $x=-1$, $f(-1)=37 \to$ absolute maximum. (global)

2) At $x=3$, $f(3)=-27 \to$ absolute minimum. (global)

3) At $x=-1$, $f'(-1)$ DNE, but there is a global max.

4) At $x=0$, local minimum & at $x=1$, local maximum.

5) We have $f'(0)=0$, $f'(1)=0$ & $f'(3)=0$.

Important observations:

a) loc max. or loc. min when $f'(x)=0$

b) max or min when $f'(x)\ \nexists$

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

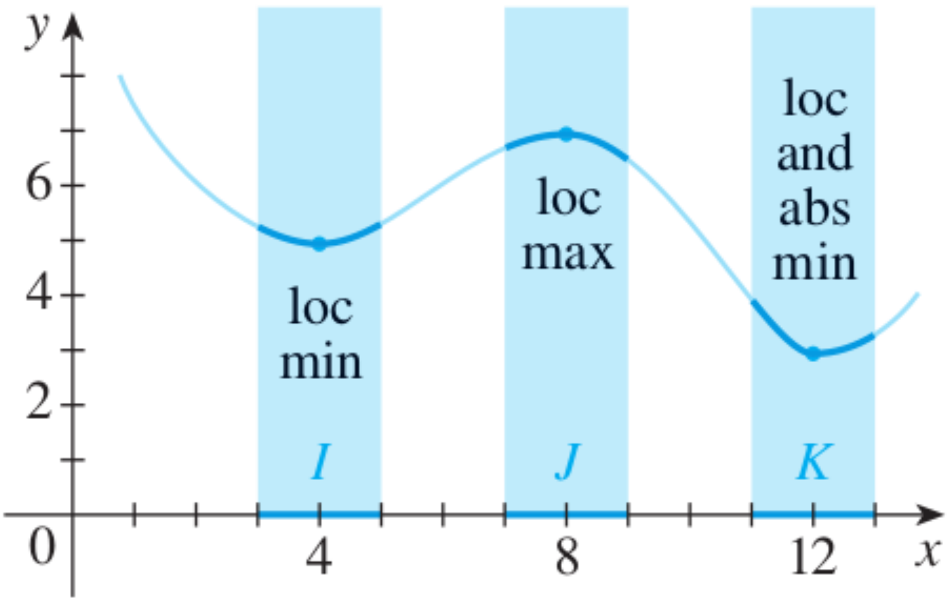


Illustration of the local and absolute max and min.

Remark: loc. max. $\not\Rightarrow$ abs. max.
abs. max. $\Rightarrow$ loc. max.

Terminology.

1) Global maximum or global minimum

2) Extreme values for abs. max. and abs. min.

Extreme Values Theorem.

Which conditions garantee that extreme values exist?

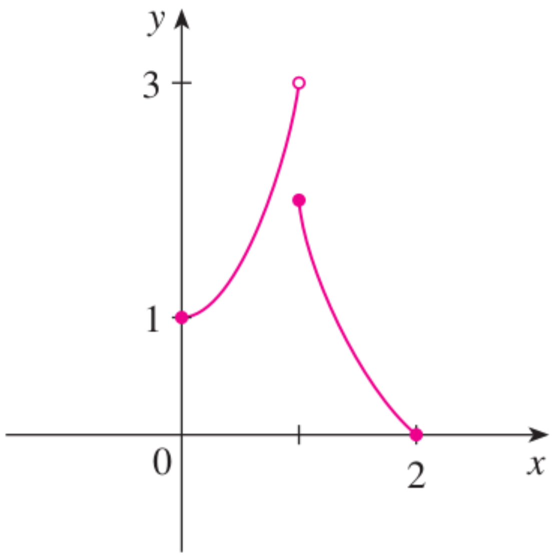


FIGURE 9
This function has minimum value $f(2) = 0$, but no maximum value.

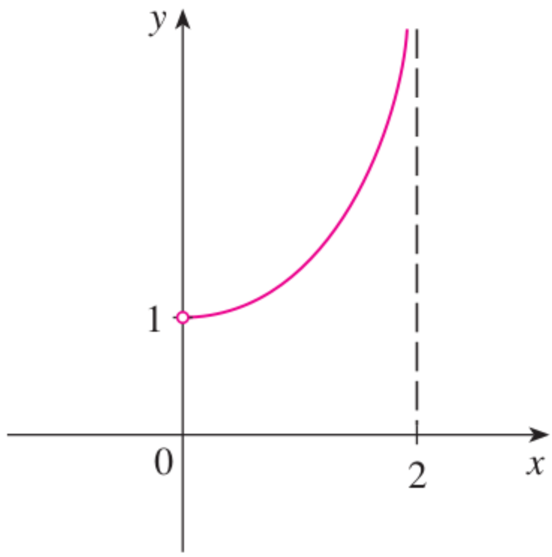
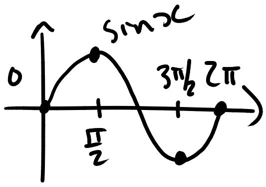
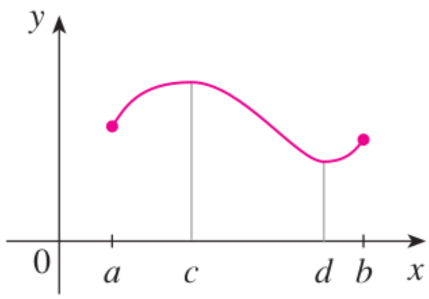


FIGURE 10
This continuous function g has no maximum or minimum.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.





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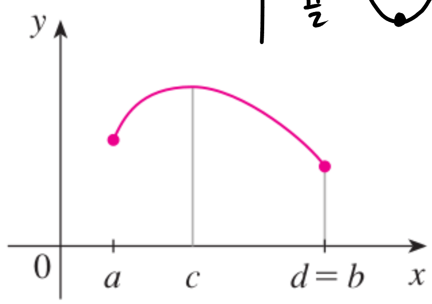

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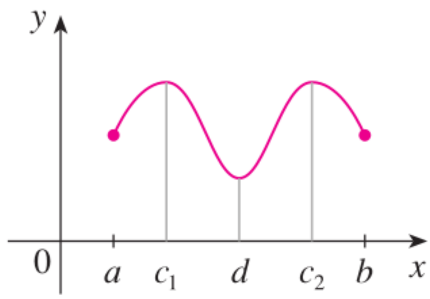
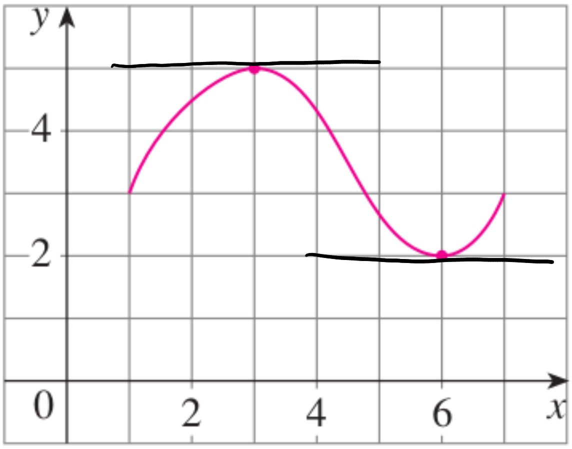


Attained multiple times

Fermat's Theorem.

An observation:

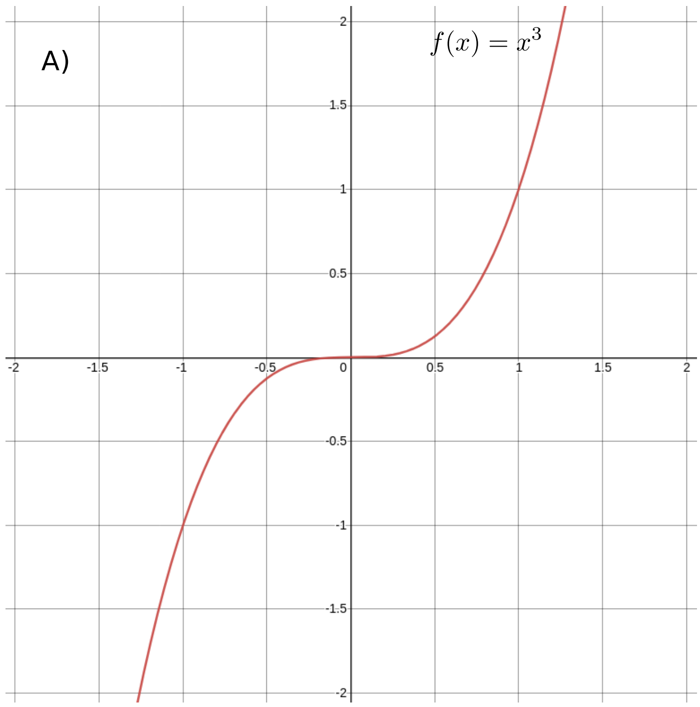
When $f(c)$ is a loc. max. or loc. min., then $f'(c)=0$.

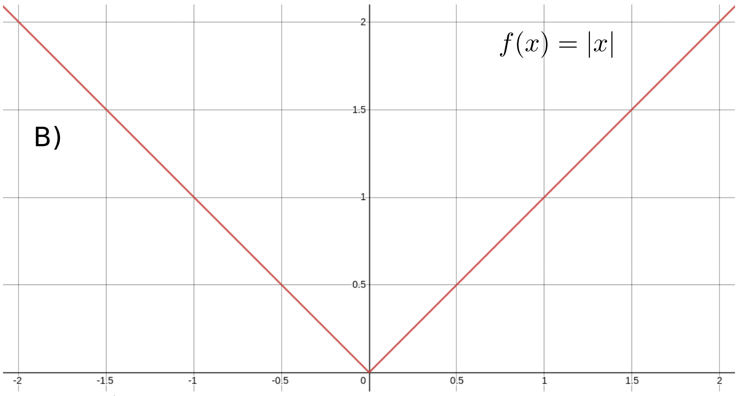


4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Interested in the proof: see page 207 in the textbook.

BE CAREFUL!!





A) $f'(x) = 3x^2 = 0 \Leftrightarrow x=0$

$f(0)$ is neither loc. max/min.

B) $f'(0)$ DNE.

but $f(0)$ is an abs. min.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

EXAMPLE 7 Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

① <u>Derivative</u>

$$f'(x) = \left(x^{3/5}\right)'(4-x) + x^{3/5}(4-x)'$$

$$= \frac{3}{5}x^{-2/5}(4-x) - x^{3/5}$$

$$= \frac{3}{5}\,\frac{4-x}{x^{2/5}} - x^{3/5}$$

$5x^{3/5} \cdot x^{2/5}$

$$= \frac{3(4-x) - 5x}{5\left(x^{2/5}\right)} = \frac{12 - 3x - 5x}{5\left(x^{2/5}\right)}$$

$$\Rightarrow \quad f'(x) = \frac{4(3-2x)}{5x^{2/5}}$$

② <u>Find the zeros of $f'(x)$</u>

$$f'(x) = 0 \iff 4(3-2x) = 0$$

$$\iff x = \frac{3}{2}$$

③ <u>Find where $f'(x)$ DNE.</u>

$f'(x)$ DNE when $x = 0$ (because $f'(0)$ explodes)

<u>Answer</u> Critical Numbers (CN) are

$$x = \frac{3}{2} \quad \text{and} \quad x = 0.$$

Finding Extremum Values on closed intervals.

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

① C.N. in $(-\frac{1}{2}, 4)$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x)$ always exists in $(-\frac{1}{2}, 4)$.

zeros: $f'(x) = 0 \iff 3x(x-2) = 0$

$\iff x = 0$ or $x = 2$.

② Evaluate f at the C.N. inside $(-\frac{1}{2}, 4)$

$f(0) = 0^3 - 3\cdot 0^2 + 1 = 1$

$f(2) = 2^3 - 3\cdot 2^2 + 1 = -3$

③ Evaluate f at the endpoints

$f(-1/2) = \frac{1}{8}$ and $f(4) = 17$

④ Answer abs max $= \max\{1, -3, \frac{1}{8}, 17\} = 17$

abs min $= \min\{1, -3, \frac{1}{8}, 17\} = -3$

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.