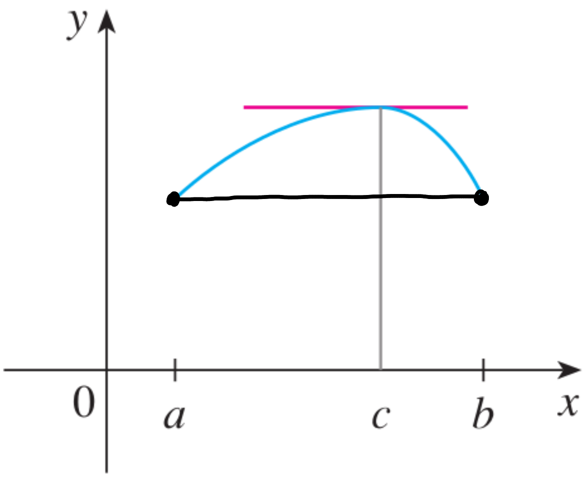
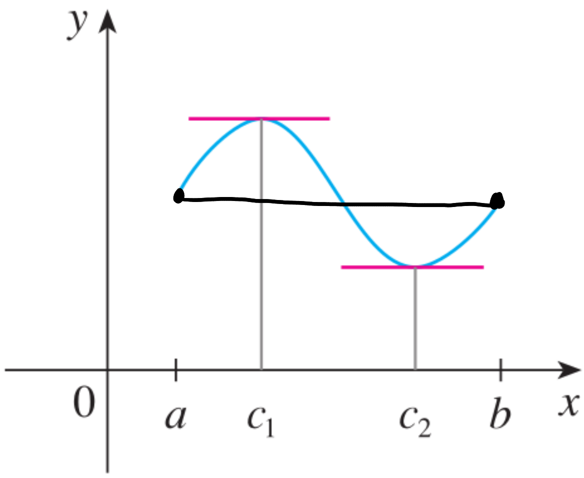
Chapter 3
Applications of Derivatives

3.2 The Mean Value Theorem

The following graphs have a commun geometric property.
Which one?



(b)



(c)

Is there a condition that garantees the graph
of a function has horizontal tangents?

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

<u>IVT:</u> $f(x) = x^3 + x - 1$.

$[a, b] = [0, 1]$.

$f(0) = -1$ and $f(1) = 1$

Here, $-1 < \underset{N}{\overset{\shortparallel}{0}} < 1 \Rightarrow$ there is a c between 0 and 1 s.t. $c^3 + c - 1 = 0$.

<u>Rolle's thm.</u> By contradiction.

Suppose there is another d s.t.

$$d^3 + d - 1 = 0.$$

Assume $d < c$.

So, $f(d) = 0$ and $f(c) = 0$

$$\Rightarrow f(d) = f(c).$$

By Rolle's Theorem, there must a a number x between d and c such that

$$f'(x) = 0.$$

We have: $f'(x) = 3x^2 + 1 \geq 1$. $\rightarrow\!\!\times\!\!\leftarrow$

This is our contradiction, so there is only one root c s.t. $c^3 + c - 1 = 0$. $\square$

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

(MVP)

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

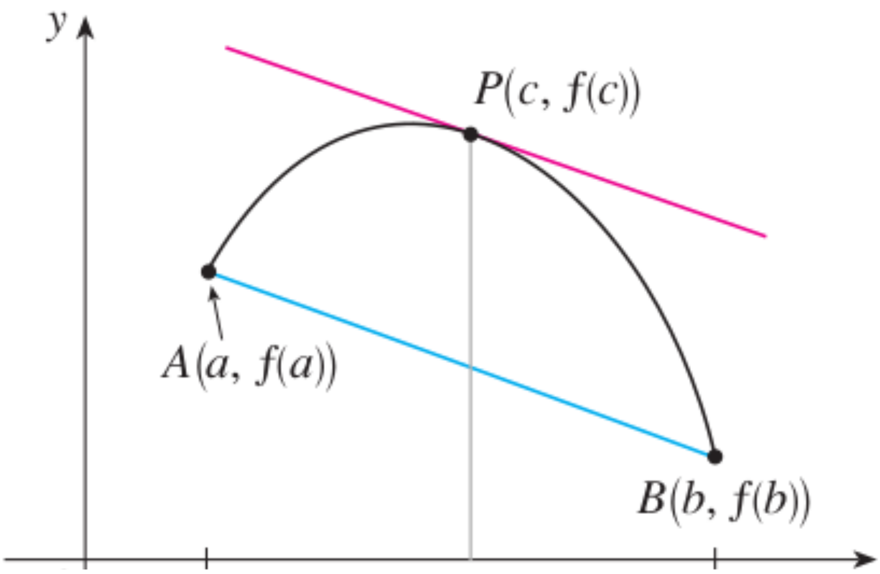
1 $$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

or, equivalently,

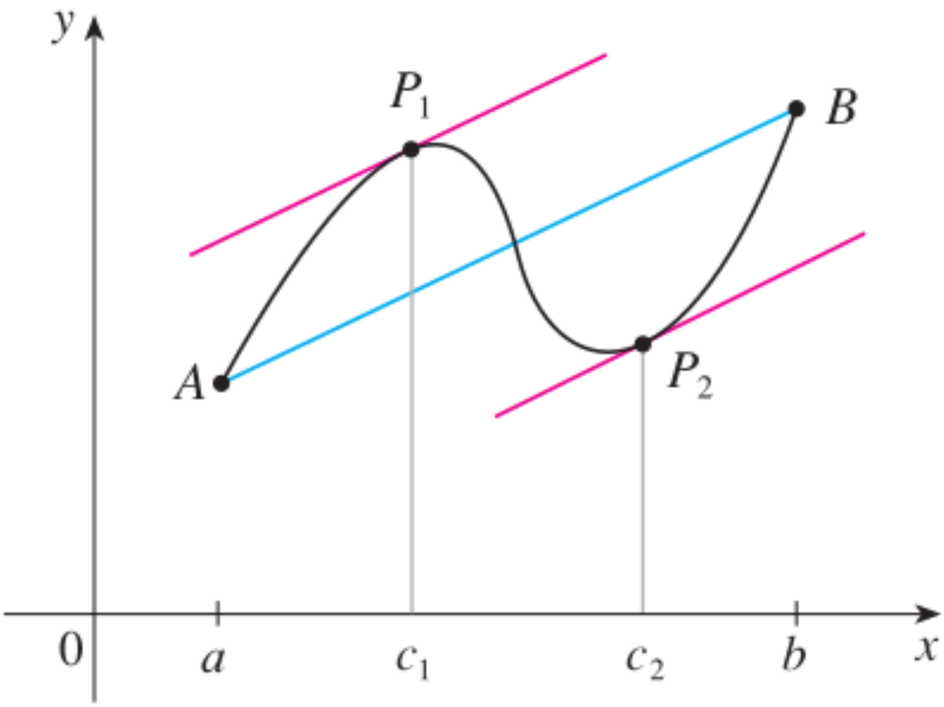
2 $$f(b) - f(a) = f'(c)(b - a)$$

The Meaning: Find a secant line with the same slope as the tangent line.

Only one c.



Multiple c.



Example

Let $f(x) = \sqrt{x}$. Find the number c that satisfies the conclusion of the Mean Value Theorem on the interval [0, 4].

$a = 0$

$b = 4$

Goal: Find c such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$f'(c) = \frac{1}{2} c^{-1/2} = \frac{1}{2\sqrt{c}}.$$

So, $$\frac{1}{2\sqrt{c}} = \frac{2 - 0}{4}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c}} = \frac{1}{2}$$

$$\Rightarrow \quad 1 = \sqrt{c}$$

$$\Rightarrow \quad \boxed{1 = c}$$

Consequences of the Mean Value Theorem.

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

Assume f is continuous and differentiable.

Take $a = 0$ and $b = 2$.

This means $\dfrac{f(2) - f(0)}{2 - 0} = f'(c)$

for some $0 < c < 2$.

So, $f(2) - (-3) = 2 \cdot f'(c)$

$\Rightarrow f(2) = 2f'(c) - 3$

So, $f(2) \leq 2 \cdot 5 - 3$ (because $f'(c) \leq 5$)

$= 7$

So, $f(2)$ can be no larger than 7.