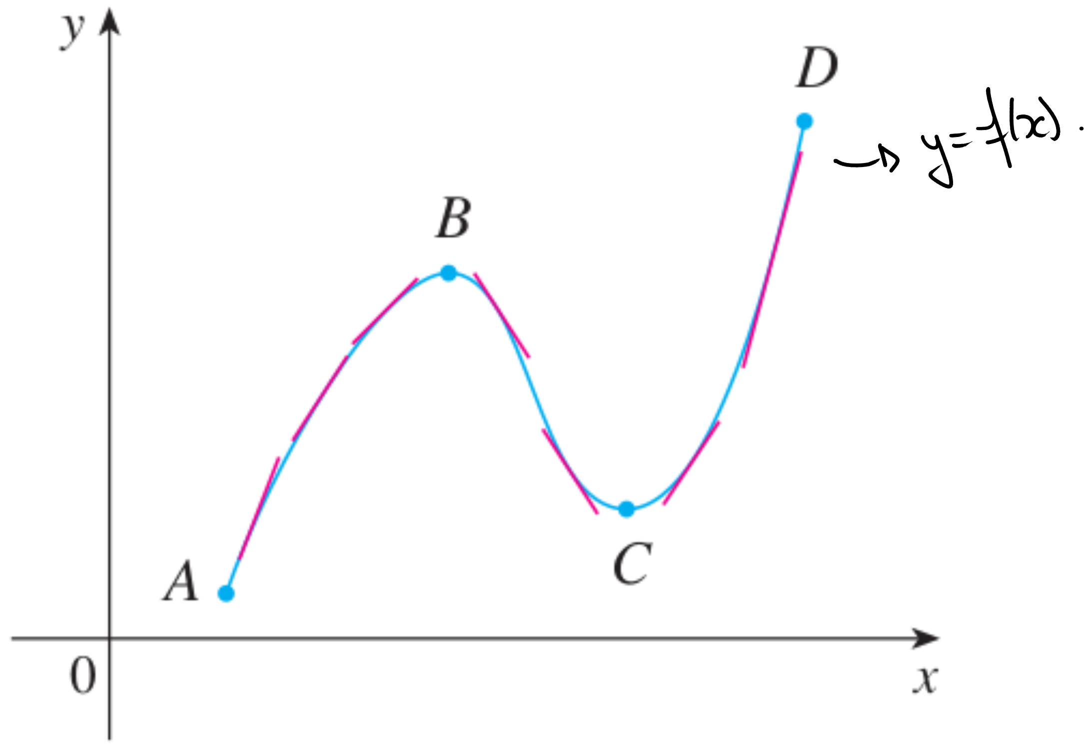
Chapter 3
Applications of Derivatives

3.3 How Derivatives affect the Shape of a Graph

What does f' tells us about f.



	A	between	B	between	C	between	D
$f'(x)$	DNE	+	0	−	0	+	DNE
$f(x)$	Abs. min.	↗	loc. max.	↘	loc. min.	↗	Abs. max.

Conclusion:

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

① Derivative $\quad f'(x) = 12x^3 - 12x^2 - 24x$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x+1)(x-2)$$

② Zeros: $\quad f'(x) = 0 \iff x = 0, \; x = -1, \; x = 2$

$x < -1$

Factors	$x <$	-1	$< x <$	0	$< x <$	2	$< x$
$x+1$	$-$	0	$+$	X	$+$	X	$+$
$x-2$	$-$	X	$-$	X	$-$	0	$+$
x	$-$	X	$-$	0	$+$	X	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	loc. min	$\nearrow$	loc max	$\searrow$	loc min	$\nearrow$

$x<-1$

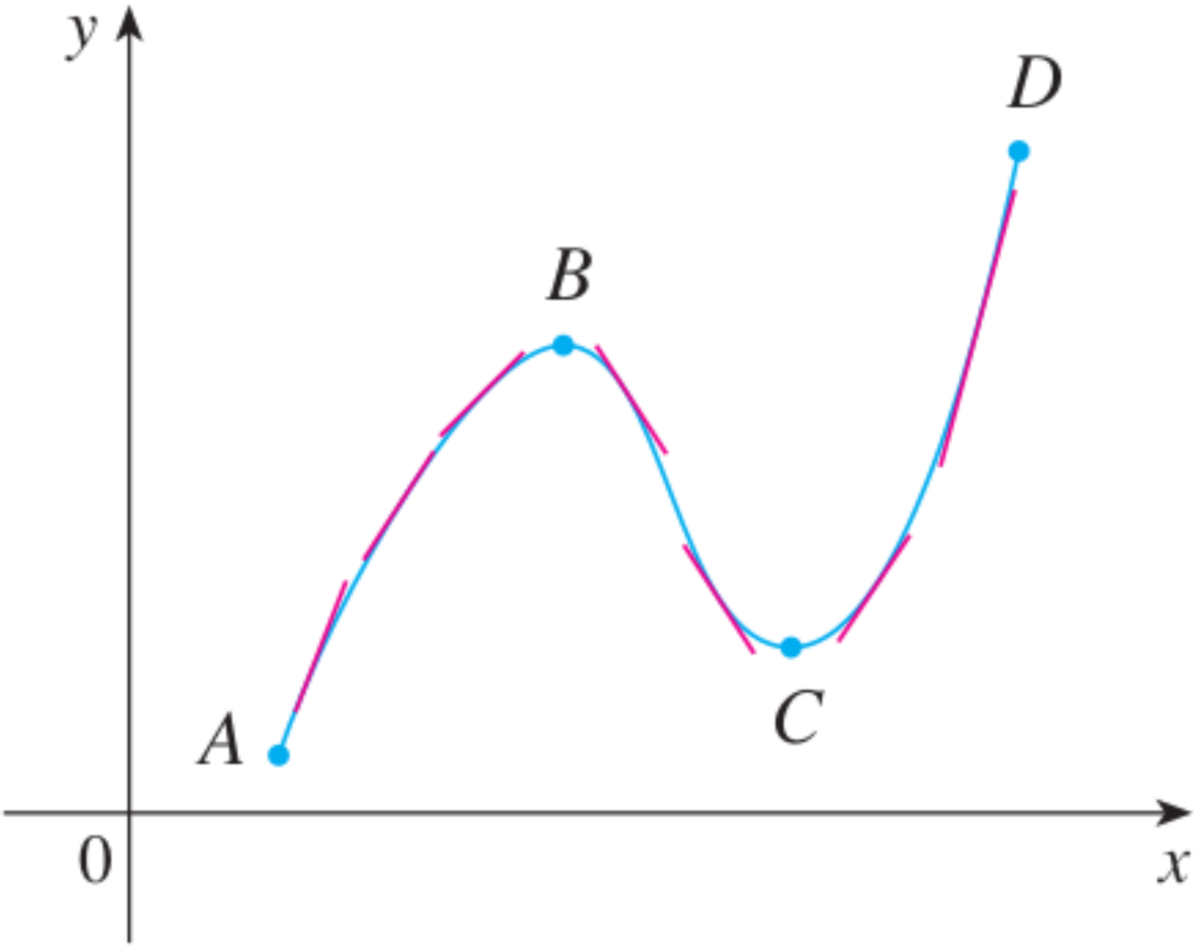
$$\begin{cases} x < -1 \to x+1 < -1+1 \to x+1 < 0 \\ x < -1 \to x-2 < -1-2 \to x-2 < -3 \\ x < -1 \end{cases}$$

$-1<x<0$

$$\begin{cases} x > -1 \to x+1 > 0 \\ -1 < x < 0 \to -1-2 < x-2 < 0-2 \to -3 < x-2 < -2 \end{cases}$$

Local Extreme Values.

CRITICAL POINTS $\not\longrightarrow$ EXTREME VALUES (MAX OR MIN)



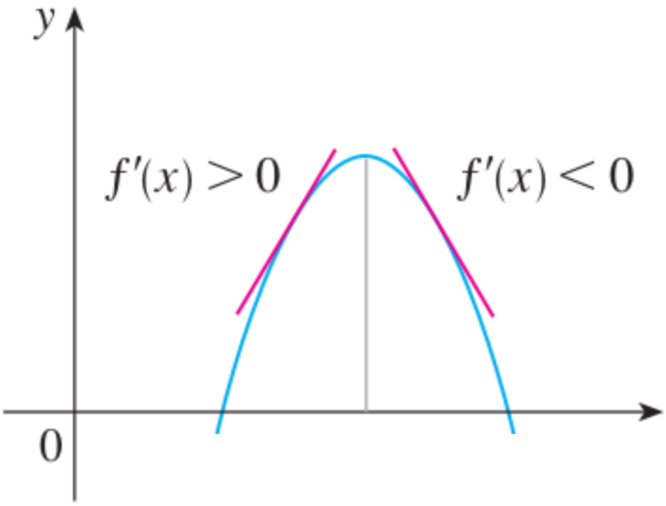
	A		B		C		D
$f'(x)$	$\nexists$	+	0	-	0	+	$\nexists$
$f(x)$	abs. min	$\nearrow$	max	$\searrow$	min	$\nearrow$	abs. max

The First Derivative Test Suppose that c is a critical number of a continuous function f.

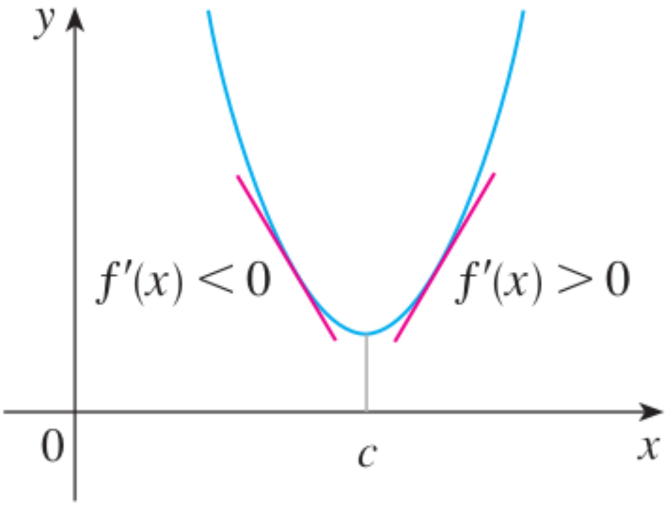
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

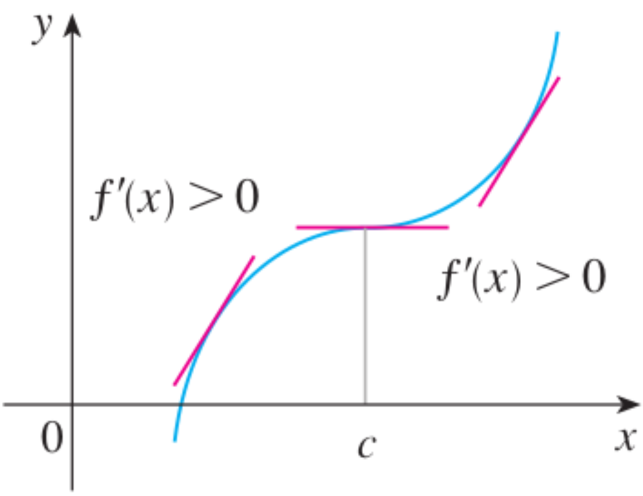
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



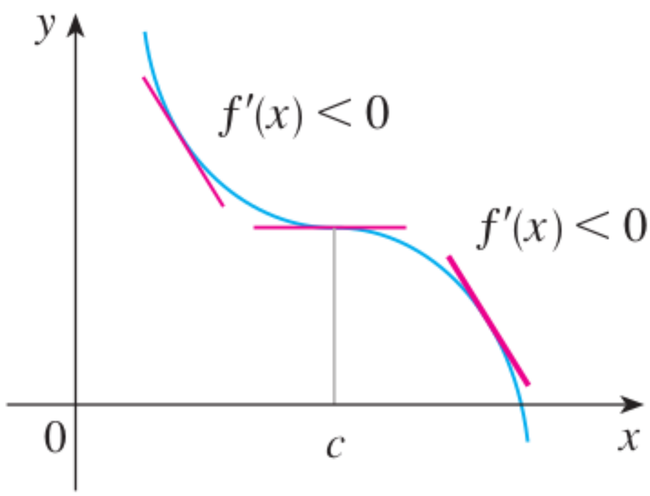
(a) Local maximum



(b) Local minimum



(c) No maximum or minimum



(d) No maximum or minimum

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

① Derivative

$$g'(x) = 1 + 2\cos x$$

zeros: $g'(x) = 0 \iff 1 + 2\cos x = 0$

$\iff \cos x = \frac{-1}{2}$

$\iff x = \frac{2\pi}{3}, \quad x = \frac{4\pi}{3}$

② Table

Factors	0	$< x <$	$\frac{2\pi}{3}$	$< x <$	$\frac{4\pi}{3}$	$< x <$	2π
$1+2\cos x$	DNE	$+$	0	$-$	0	$+$	DNE
$f(x)$		$\nearrow$	loc max.	$\searrow$	loc. min	$\nearrow$	

$x = \pi/2$

$\rightarrow 1 + 2\cos(\frac{\pi}{2}) = 1 > 0$

$x = \pi$

$\rightarrow 1 + 2\cos\pi = -1 < 0$

$x = \frac{3\pi}{2}$

$1 + 2\cos(\frac{3\pi}{2}) = 1 > 0$

③ Min & max. values.

At $x = \frac{2\pi}{3}$ $\quad f(\frac{2\pi}{3}) = \frac{2\pi}{3} + 2\sin(\frac{2\pi}{3})$

$= \frac{2\pi}{3} + \frac{2\sqrt{3}}{2} = \boxed{\frac{2\pi}{3} + \sqrt{3}}$

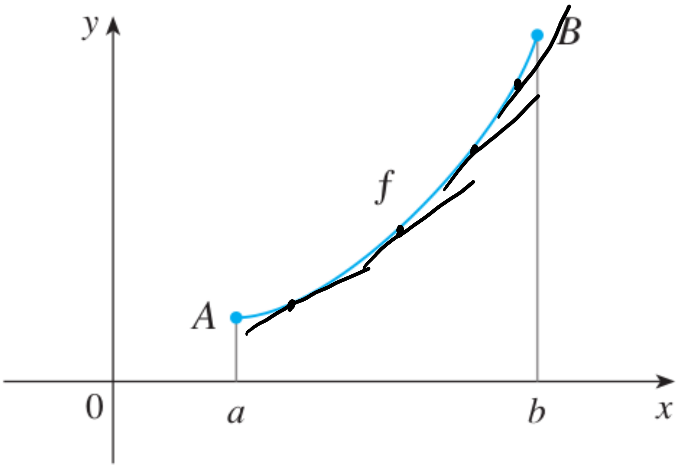
At $x = \frac{4\pi}{3}$ $\quad f(\frac{4\pi}{3}) = \boxed{\frac{4\pi}{3} - \sqrt{3}}$

What does f'' tell us about f?

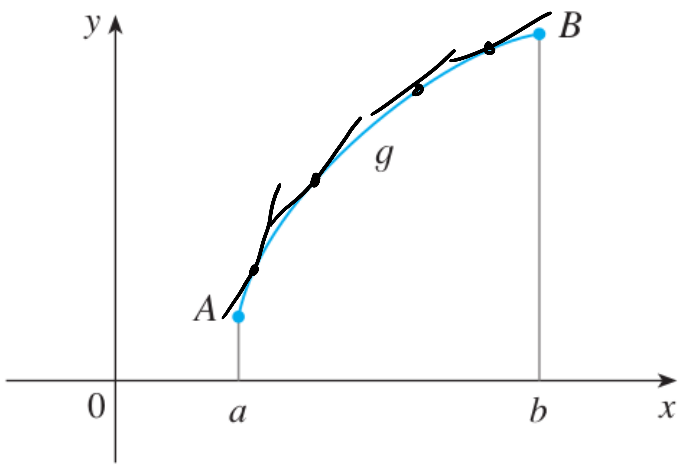
Two important definitions:

1) **Definition** If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

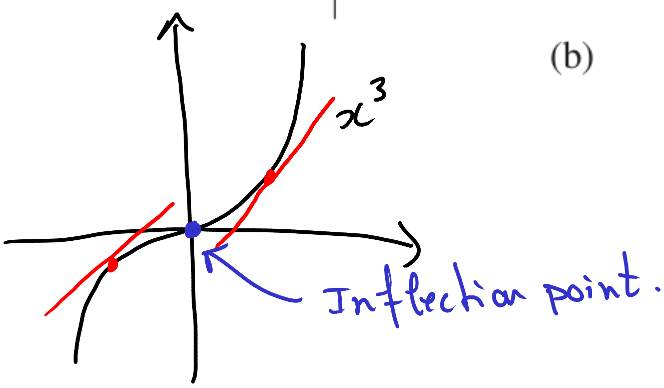
2) **Definition** A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



(a)



(b)



Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Note: There is an inflection point when the second derivative is zero.

or DNE.

Example. Find the interval(s) of concavity of the function

$$f(x) = x^3 - 3x^2 - 9x + 4$$

① 2nd derivative.

$$f'(x) = 3x^2 - 6x - 9$$

$$f''(x) = 6x - 6 = 6(x-1)$$

② Zeros

$$f''(x) = 0 \iff 6x = 6$$

$$\iff x = 1$$

③ Draw.

Factors	$-\infty < x <$	1	$< x < \infty$
$x-1$	$-$	0	$+$
$f''(x)$	$-$		$+$
$f(x)$			

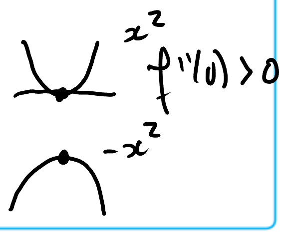
$x < 1$, $f(x)$ is concave down.

$x > 1$, $f(x)$ is concave up.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



REMARK!

- $f''(x) > 0$ for any x and $f'(c)=0$ $\Rightarrow$ c is an absolute minimum.
- $f''(x) < 0$ for any x and $f'(c)=0$. $\Rightarrow$ c is an absolute max.
- $f''(c) = 0$ $\Rightarrow$ can't conclude

EXAMPLE. Find the extreme values of the function $f(x) = x^3 + 3x^2$.

① Critical numbers.

$$f'(x) = 3x^2 + 6x = 3x(x+2)$$

C.N.: $f'(x) = 0 \Leftrightarrow x = 0$ or $x = -2$.

② 2^{nd} derivative.

$$f''(x) = 6x + 6 = 6(x+1)$$

$x=0$ $\quad f''(0) = 6(0+1) = 6 > 0$

$\rightarrow$ $x=0$ is a local minimum.

$\rightarrow$ $f(0) = 0$ (loc. min. value).

$x=-2$ $\quad f''(-2) = 6(-2+1) = -6 < 0$

$\rightarrow$ $x=-2$ is a local max.

$\rightarrow$ $f(-2) = 4$ (loc. max. value).