Chapter 3
Applications of Derivatives

3.4 Limits at Infinity; Horizontal Asymptotes

Example. What is the limit of $f(x) = \dfrac{x^2-1}{x^2+1}$ when x becomes large?

x	$f(x)$
10	≈ 0.99
100	≈ 0.9998
1000	≈ 0.999998

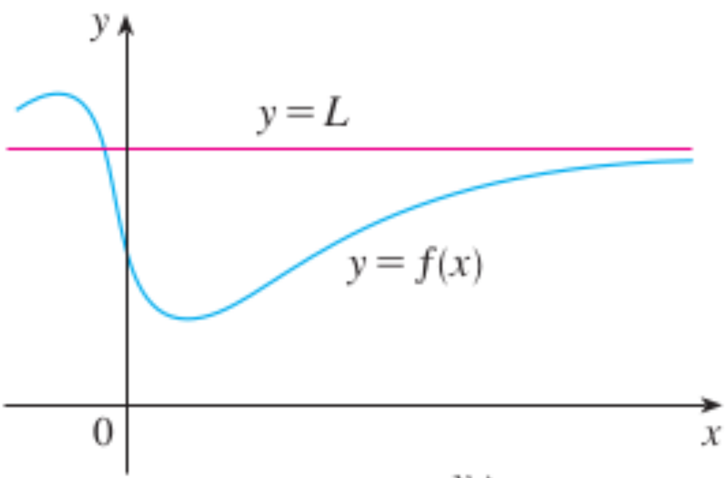
x	$f(x)$
10000	≈ 0.99999998
100 000	≈ 0.9999999998
$\vdots$	$\vdots$
$\downarrow$ ∞	$\downarrow$ 1

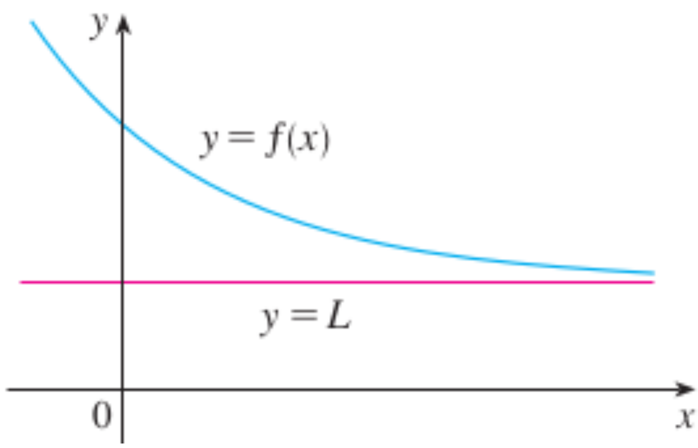
$$\lim_{x \to \infty} \frac{x^2-1}{x^2+1} = 1$$

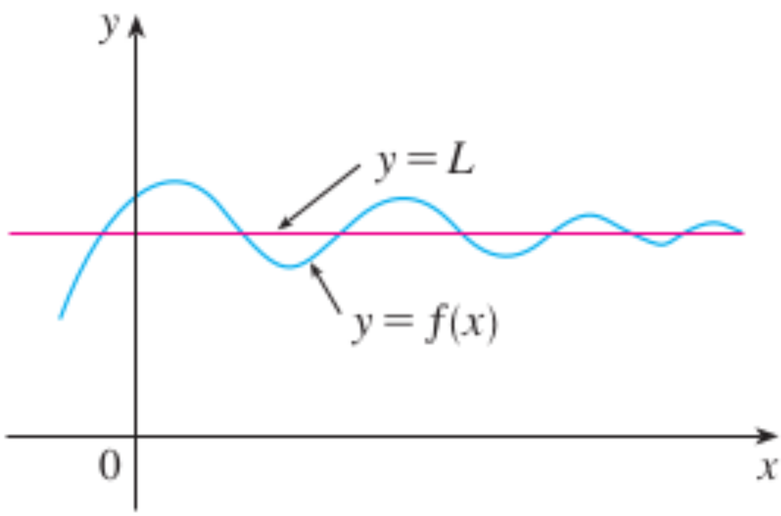
1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.







Example. What is the limit of $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ when x becomes large?

x	$f(x)$
-10	≈ 0.99
$\vdots$	
-10000	≈ 0.99999998

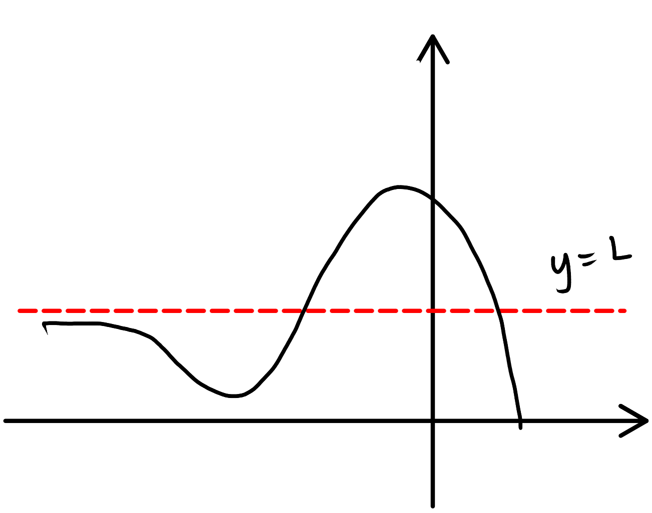
x	$f(x)$
$-100\,000$	0.9999999998
$\vdots$	$\downarrow$
$\downarrow$ ∞	1

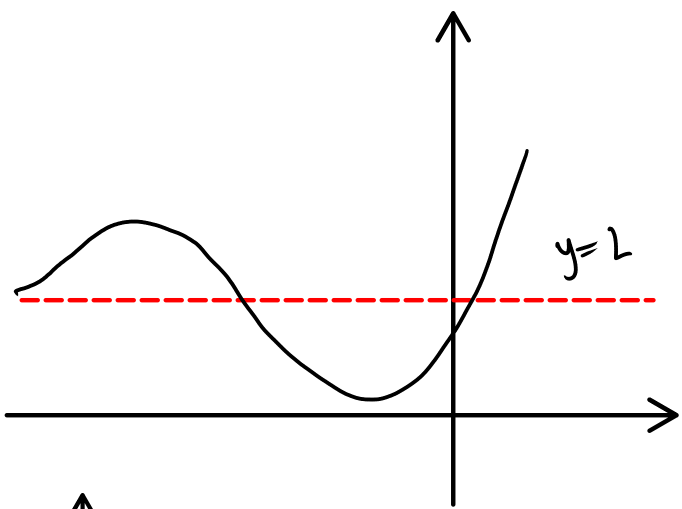
$$\lim_{x \to -\infty} \frac{x^2 + 1}{x^2 - 1} = 1$$

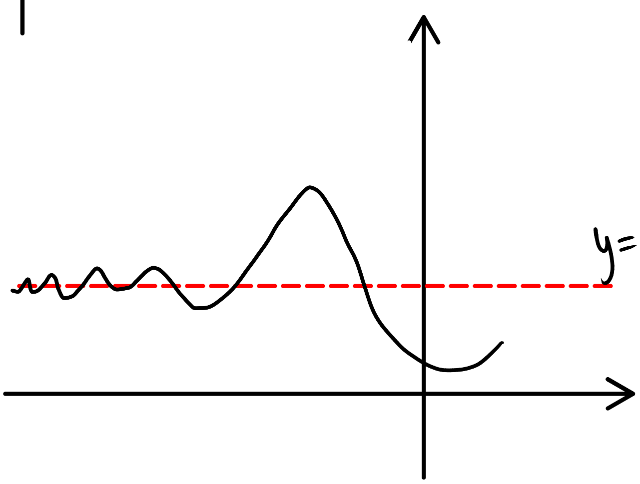
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.







3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

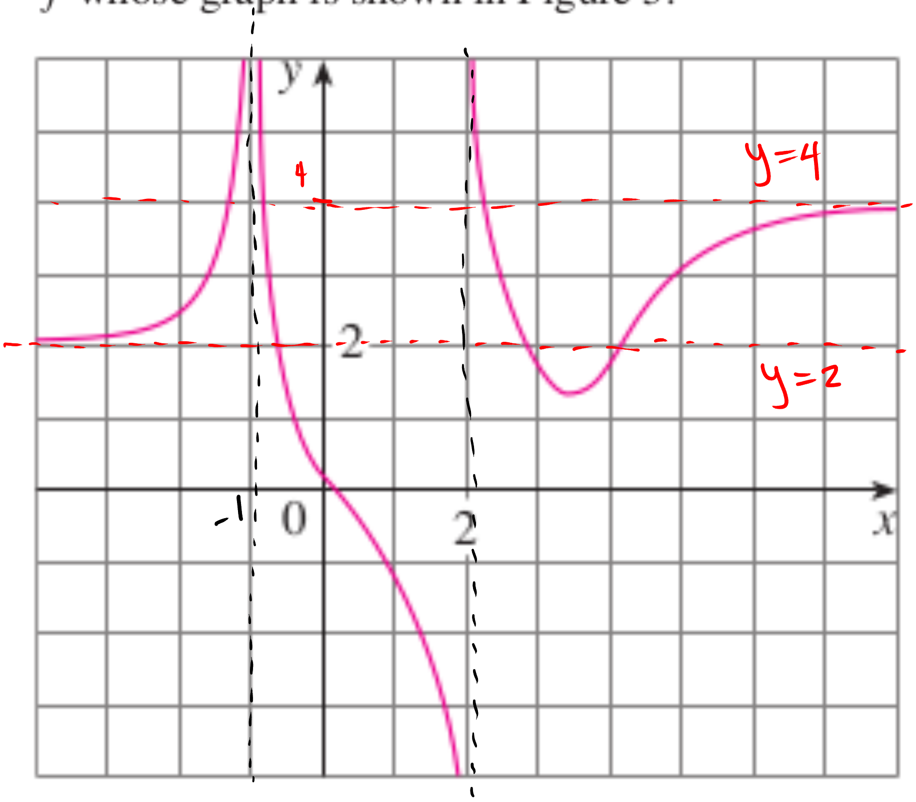


FIGURE 5

A) Infinite limits.

$$\lim_{x \to -1} f(x) = +\infty$$

$$\lim_{x \to 2^-} f(x) = -\infty$$

and

$$\lim_{x \to 2^+} f(x) = +\infty$$

$x = -1$ is a VA

$x = 2$ is a VA.

B) limists at infinity

$$\lim_{x \to -\infty} f(x) = 2.$$

$$\lim_{x \to \infty} f(x) = 4$$

$y = 2$ HA

$y = 4$ HA.

Rules for Limits at infinity.

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

e.g. $r = 1/3$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

① no $r \leq 0$

② $\lim_{x\to-\infty} \frac{1}{x^r}$ is not defined for all $r>0$.

EXAMPLE 3 Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

highest power are the same, the limit is the quotient of the leading coef.

Factor higher power of x:

$$\frac{3x^2-x-2}{5x^2+4x+1} = \frac{\cancel{x^2}(3 - x^{-1} - 2x^{-2})}{\cancel{x^2}(5 + 4x^{-1} + x^{-2})} = \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

① $\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right) = \lim_{x\to\infty} 3 - \underbrace{\cancel{\lim_{x\to\infty}\frac{1}{x}}}_{0} - 2\underbrace{\cancel{\lim_{x\to\infty}\frac{1}{x^2}}}_{0}$

$= 3 - 0 - 2\cdot 0 = 3$

② $\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right) = \lim_{x\to\infty} 5 + 4\underbrace{\cancel{\lim_{x\to\infty}\frac{1}{x}}}_{0} + \underbrace{\cancel{\lim_{x\to\infty}\frac{1}{x^2}}}_{0}$

$= 5 + 4\cdot 0 + 0 = 5$

$$\lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{\lim_{x\to\infty} 3 - \frac{1}{x} - \frac{2}{x^2}}{\lim_{x\to\infty} 5 + \frac{4}{x} + \frac{1}{x^2}} = \boxed{\frac{3}{5}}$$

When highest powers are different:

$$\frac{x^3+x}{x^2+x} = \frac{x^3\left(1+\frac{1}{x^2}\right)}{x^2\left(1+\frac{1}{x}\right)}$$

$\cancel{\lim_{x\to\infty}(\ldots) \overset{?}{=} 1}$

$$= x\left(\frac{1+\frac{1}{x^2}}{1+\frac{1}{x}}\right)$$

$$\lim_{x\to\infty}\frac{x^3+x}{x^2+x} = \underbrace{\left(\lim_{x\to\infty} x\right)}_{+\infty}\underbrace{\left(\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1+\frac{1}{x}}\right)}_{\frac{1}{1}}$$

$$= +\infty.$$

EXAMPLE 4 Find the horizontal ~~and vertical asymptotes~~ of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

$$f(x) = \frac{\sqrt{x^2\left(2 + \frac{1}{x^2}\right)}}{x\left(3 - \frac{5}{x}\right)} = \frac{\sqrt{x^2}\,\sqrt{2 + \frac{1}{x^2}}}{x\left(3 - \frac{5}{x}\right)}.$$

[Note: $\sqrt{x^2} = \sqrt{4} \Rightarrow \sqrt{x^2} = 2 \Rightarrow |x| = 2$]

So, $\sqrt{x^2} = |x|$

HA: $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{|x|\sqrt{2 + \frac{1}{x^2}}}{x\left(3 - \frac{5}{x}\right)}$$ (Note: $x > 0$) $|x| = x$

$$= \lim_{x \to \infty} \frac{\cancel{x}}{\cancel{x}} \frac{\sqrt{2 + \frac{1}{x^2}}}{\left(3 - \frac{5}{x}\right)}$$

$$\boxed{y = \frac{\sqrt{2}}{3} \text{ is HA.}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{\left(3 - \frac{5}{x}\right)} = \frac{\sqrt{2}}{3}$$

$$\lim_{x \to -\infty} \frac{|x|\sqrt{2 + \frac{1}{x^2}}}{x\left(3 - \frac{5}{x}\right)} = \lim_{x \to -\infty} \frac{-\cancel{x}\sqrt{2 + \frac{1}{x^2}}}{\cancel{x}\left(3 - \frac{5}{x}\right)}$$

$$\boxed{y = -\frac{\sqrt{2}}{3} \text{ is HA.}}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = \frac{-\sqrt{2}}{3}$$

EXAMPLE 5 Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$. $\longrightarrow$ $\infty - \infty$

<u>Simplify:</u>

$$\left(\sqrt{x^2+1}-x\right)\cdot\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)} = \frac{1}{\sqrt{x^2+1}+x}$$

$$\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}+x}$$

$$= \lim_{x\to\infty}\frac{1}{x\sqrt{1+\frac{1}{x^2}}+x}$$

$$= \lim_{x\to\infty}\frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)}$$

$$= \left(\lim_{x\to\infty}\frac{1}{x}\right)\left(\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}+1}\right)$$

$$= 0\cdot\frac{1}{2} = 0.$$

Infinite Limits at Infinity.

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

means that the values of $f(x)$ become larger and larger as the values of x becomes larger and larger. Similar meanings are attached to the following symbols:

$$\lim_{x\to-\infty} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

WARNING!!

undefined $\leftarrow$ $\infty - \infty \neq 0$

EXAMPLE 8 Find $\lim_{x\to\infty} x^3$ (1) and $\lim_{x\to-\infty} x^3$ (2).

(1) $x = 10$ $\qquad\qquad$ $x = 1000$

$\Rightarrow x^3 = 10^3 = 1000$ $\qquad\qquad$ $\Rightarrow x^3 = 1000^3 = 10^9$

x gets bigger and bigger, x^3 gets bigger and bigger

$$\Rightarrow \lim_{x\to\infty} x^3 = \infty$$

(2) $x = -10$ $\qquad\qquad$ $x = -1000$

$\Rightarrow x^3 = (-10)^3 = (-1)^3(10)^3$
$= -10^3$

$\Rightarrow x^3 = -(1000)^3$
$= -10^9$

$$\Rightarrow \lim_{x\to-\infty} x^3 = -\infty.$$

General: $r > 0$, $\lim_{x\to\infty} x^r = \infty$

EXAMPLE 9 Find $\lim_{x\to\infty} (x^2 - x)$. $\boxed{\infty - \infty}$ Undefined.

$$\lim_{x\to\infty} x^2 - x = \lim_{x\to\infty} x(x-1)$$

$$= \lim_{x\to\infty} x \lim_{x\to\infty} x-1 = \infty$$

$$\infty \cdot \infty$$

Rules with infinities:

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} g(x) = -\infty$$

and $\lim_{x\to\infty} h(x) = \infty$.

- $\lim_{x\to\infty} f(x)h(x) = \infty \quad (= \infty \cdot \infty)$
- $\lim_{x\to\infty} (f(x)+c) = \infty \quad (= \infty + c)$, c real number.
- $\lim_{x\to\infty} (f(x)-c) = \infty \quad (= \infty - c)$, c real number
- $\lim_{x\to\infty} g(x) f(x) = -\infty \quad (= (-\infty)\cdot \infty)$
- $\lim_{x\to\infty} (g(x)+c) = -\infty \quad (= -\infty + c)$, c real number
- $\lim_{x\to\infty} (g(x)-c) = -\infty \quad (= -\infty - c)$, c real number.