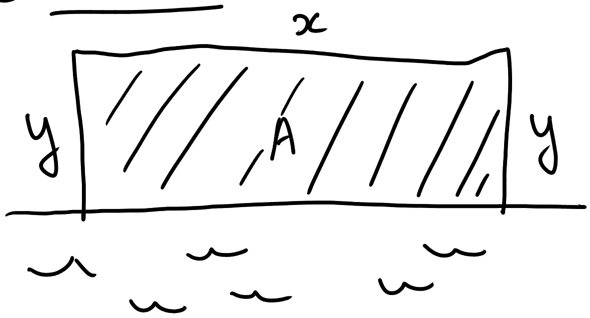
Chapter 3
Applications of Derivatives

3.7 Optimization Problems

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area? https://www.desmos.com/calculator/35rqnrbdmh

① Sketch



② Notations.

x: width of the field (ft)

y: heigth of the field (ft)

A: Area of the field (ft^2)

③ Rule\ Equation

$$A = xy.$$

④ Eliminate one of the variable

total (Amount) of fencing = 2400

$$\Rightarrow \quad y + x + y = 2400 \quad \Rightarrow \quad 2y + x = 2400$$

$$\Rightarrow \quad x = 2400 - 2y$$

So, $A = (2400 - 2y)\,y = 2400y - 2y^2.$

⑤ Optimize.

$$\frac{dA}{dy} = 2400 - 4y = 0$$

$$\Leftrightarrow \quad 2400 = 4y$$

$$\Leftrightarrow \quad 600 = y$$

Domain function: $[0, 1200]$

5.1 Closed interval method:

$A(0) = 0, \quad A(1200) = 0, \quad A(600) = 720\,000$

So, max area $= 720\,000 \text{ ft}^2$.

when $y = 600 \text{ ft}$.

5.2 2nd derivative test:

$$\frac{d^2A}{dy^2} = -4 < 0 \quad \text{for any value of } y.$$

So, $A(600) = 720\,000$ is an absolute max by the 2nd derivative test.

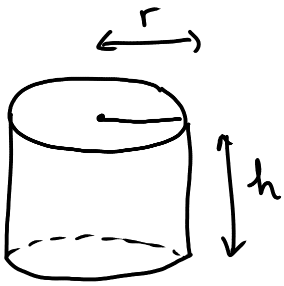
Answer:

$$x = 2400 - 2\underset{=600}{y} = 1200 \text{ ft}$$

$$y = 600 \text{ ft}$$

$$A = 720\,000 \text{ ft}^2$$

EXAMPLE 2 A cylindrical can is to be made to hold 1 L (1000 cm^3) of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

① 

② r: radius (cm)

h: heigth (cm)

V: volume (cm^3)

A: surface area (cm^2)

③ $A = 2 \times A(\bigcirc_r) + 1 \times A\left(h\,\boxed{\quad 2\pi r \quad}\right)$

$= 2\pi r^2 + 2\pi r h$.

④ Volume $= V = 1000\ cm^3$

$\Rightarrow \pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2}$

So, $A = 2\pi r^2 + 2\cancel{\pi} r\left(\dfrac{1000}{\cancel{\pi} r^2}\right)$

$\Rightarrow A = 2\pi r^2 + \dfrac{2000}{r}$, $r > 0$.

$2000 r^{-1}$
$\hookrightarrow -2000 r^{-2}$

⑤ $\dfrac{dA}{dr} = 4\pi r + \left(-\dfrac{2000}{r^2}\right) = 4\pi r - \dfrac{2000}{r^2}$

C.N: $4\pi r - \dfrac{2000}{r^2} = 0 \iff 4\pi r = \dfrac{2000}{r^2}$

$\iff r^3 = \dfrac{500}{\pi}$

$\iff r = \sqrt[3]{\dfrac{500}{\pi}}$

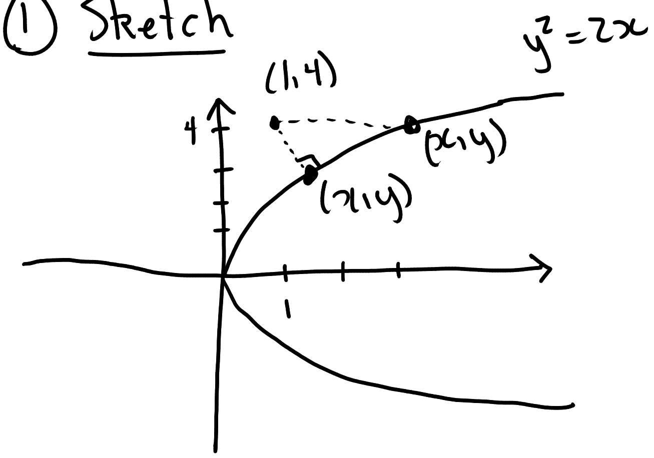
$\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{4000}{r^3} > 0$ for any values of $r > 0$.

By the 2^{nd} derivative test, $r = \sqrt[3]{\frac{500}{\pi}}$ is a minimum (abso absolute).

$$\boxed{\begin{aligned} &\underline{\text{Answer:}} \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.419 \text{ cm} \\ &h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} \approx 10.839 \text{ cm}. \end{aligned}}$$

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

① Sketch



② Notations

(x,y): point on the graph

d: distance from $(1,4)$ to (x,y)

Goal: Closest point (x,y) to $(1,4)$

③ Function

$$\text{Distance} = \sqrt{(1-x)^2 + (4-y)^2}$$

Make things easier: square the distance function

(That's the trick)

$$\Rightarrow \quad D = (1-x)^2 + (4-y)^2$$

④ Elimination/Function to optimize

We know (x,y) should be on the graph,

so $y^2 = 2x \Rightarrow x = \frac{y^2}{2}$

Therefore

$$D = \left(1 - \frac{y^2}{2}\right)^2 + (4-y)^2.$$

Domain: $y > 0$.

⑤ Optimize.

$$\frac{dD}{dy} = -2y\left(1 - \frac{y^2}{2}\right) - 2(4-y)$$

$$= -2y + y^3 - 8 + 2y$$

$$= y^3 - 8.$$

So, $\frac{dD}{dy} = 0 \iff y^3 = 8$

$\iff y = 2$

<u>2^{nd} derivative:</u> $\frac{d^2D}{dy^2} = 3y^2 > 0$ if $y > 0$.

$\Rightarrow$ $y = 2$ corresponds to an abs. min.

<u>Answer</u>: $x = \frac{y^2}{2} = \frac{4}{2} = 2$

$y = 2$

$d = \sqrt{5}$