Chapter 3
Applications of Derivatives

3.8 Newton's Method

Roots of polynomials.

- for quadratic polynomial $f(x) = ax^2 + bx + c$, the roots are given by:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{a}$$

- There are formulas for cubics and quartics (horribly long...).
  - -> cubic: https://encyclopediaofmath.org/wiki/Cardano_formula
  - -> quartic: https://encyclopediaofmath.org/wiki/Ferrari_method

- For polynomials of degree greater than 4, there is no general formula!
  Interested in the proof: http://www.math.caltech.edu/~jimlb/abel.pdf

Niels Henrik Abel

- 1802-1829
- Died from Turberculosis

Evariste Galois

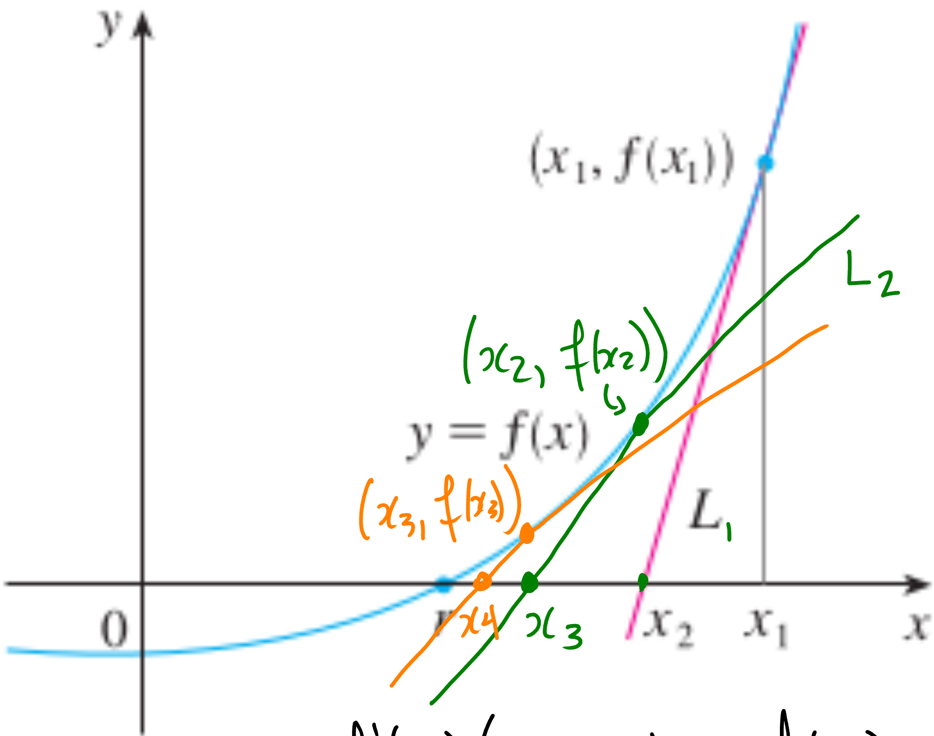
- Died in a duel for a mysterious mistress...
- 1811-1832

<u>The urgent need of Newton's method!</u>

KEY IDEAS:

- The tangent line approximate well the function.
- Replace the fonction with its tangent line.
- Intersect the tangent line with the x-axis.

Data:



<u>Eq. tangent</u>: $y = f'(x_1)(x - x_1) + f(x_1)$

<u>Intersection</u> $0 = f'(x_1)(x - x_1) + f(x_1)$

$$\Rightarrow \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad (f'(x_1) \neq 0)$$

<u>Intersection</u>: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} \quad (f'(x_2) \neq 0)$

Iterate (repeat) the process.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $\underbrace{\frac{x^3}{2} - 3x}_{=f(x)} = 0$.

Step 1 $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(x) = \frac{x^3}{2} - 3x$

$$= 2 - \frac{f(2)}{f'(2)}$$

$f'(x) = \frac{3}{2}x^2 - 3$

$$\Rightarrow \quad x_2 = 2 - \frac{(-2)}{3} = \frac{8}{3} \approx 2.67$$

Step 2 $\quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= \frac{8}{3} - \frac{f(8/3)}{f'(8/3)}$$

$$\Rightarrow \quad x_3 = \frac{8}{3} - \frac{40/27}{69/9} = \frac{512}{207}$$

$$\approx 2.4734.$$

n	xn
1	2
2	2.666666667
3	2.473429952
4	2.44983289
5	2.449489815
6	2.449489743
7	2.449489743
8	2.449489743
9	2.449489743
10	2.449489743
11	2.449489743
12	2.449489743
13	2.449489743
14	2.449489743
15	2.449489743

Newton's Method May Fail

Take a look at the formula in Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where do you think this formula might fail?

- Problem when $f'(x_n) \approx 0$.
- x_{n+1} might be outside of the domain.

Example. Redo the last example with $x_1 = -1.14$.

Desmos: https://www.desmos.com/calculator/nm3bpdg95t

$f(x) = \frac{x^3}{2} - 3x \rightarrow f'(x) = \frac{3}{2}x^2 - 3$.

- Step 1: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.41$
- Step 2: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx -164.4512\ldots$

$MANY^{MANY} APPLICATIONS$!!!

- Finding solutions to general equations such as

$$\cos(x) = x$$

- At the core of many numerical methods in ingeneering.

- Generate wonderful fractal pictures:
  Watch the 3blue1brown video
  https://www.youtube.com/watch?v=-RdOwhmqP5s