Math 241

Chapter 3

Section 3.9: Antiderivatives

Contents

Created by: Pierre-Olivier Parisé

Spring 2023

Definition

A function F is an **antiderivative** of a function f if $F'(x) = f(x)$.

Example 1. Find an antiderivative of the following functions.

(a) $f(x) = x^2$. (b) $g(x) = 3x^3 + \cos(x)$. (c) $h(x) = x^{2/3} + 4\sec^2(x)$.

(a) $F(x) = \frac{1}{3}x^3 \xrightarrow{'} F'(x) = \frac{1}{3}(3x^2) = x^2$

$F(x) = \frac{1}{3}x^3 + 2. \xrightarrow{'} F'(x) = \frac{1}{3}(3x^2) + 0 = x^2$

(b) $G(x) = \frac{3}{4}x^4 + \sin(x) + 2 \xrightarrow{'} G'(x) = \frac{3}{\cancel{4}}(\cancel{4}x^3) + \cos x + 0$

$= 3x^3 + \cos x$

(c) $H(x) = \frac{3}{5}x^{\frac{5}{3}} + 4\tan(x)$

$\xrightarrow{'} H'(x) = \frac{\cancel{3}}{\cancel{5}}\left(\frac{\cancel{5}}{\cancel{3}}x^{2/3}\right) + 4\sec^2 x + 0$

$= x^{2/3} + 4\sec^2 x$

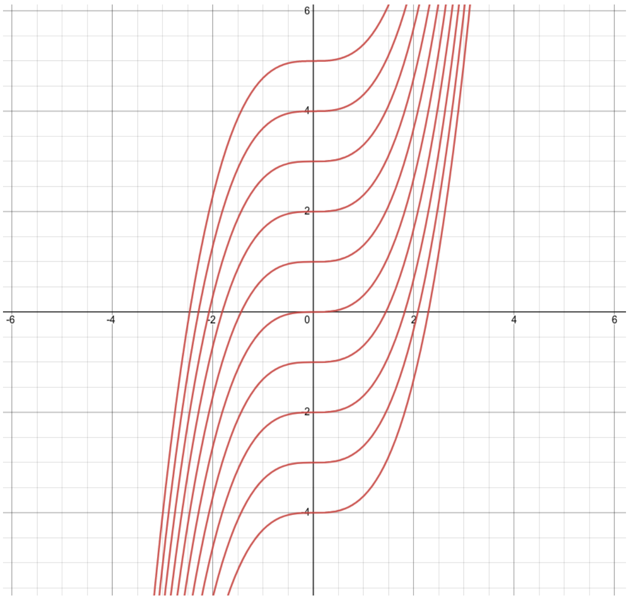
Remarks:

- Recall that $f'(x) = g'(x)$ if and only if $f(x) = g(x) + C$ for some constant C.
- There are more than just one antiderivative!

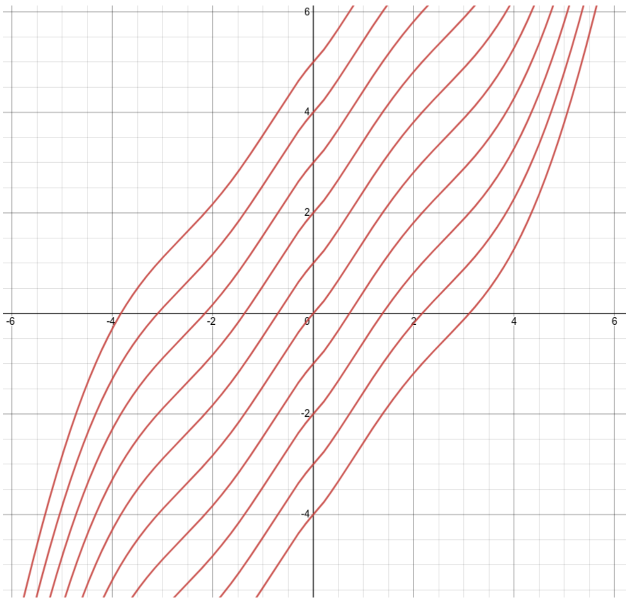
The **most general antiderivative** of a function f is

$$F(x) + C,$$

where C is a constant.



(a) Several Antiderivatives of $f(x) = x^2$, that is $\frac{x^3}{3} + C$



(b) Several antiderivatives of $f(x) = x^{2/3} + \cos(x)$, that is $\frac{3}{5}x^{5/3} + \sin(x) + C$.

Example 2. Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$. **(b)** $f(x) = x^n$, $n \geq 0$.

(a) $F(x) = -\cos x \longrightarrow F'(x) = -(-\sin x) = \sin x$.

Gen. Anti. $= \boxed{-\cos x + C}$

(b) $F(x) = \frac{1}{n+1} x^{n+1} \longrightarrow F'(x) = \frac{n+1}{n+1} x^n = x^n$.

Gen. Anti $= \boxed{\frac{1}{n+1} x^{n+1} + C}$

Table of Antiderivatives

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

Figure 2: Properties and some Antiderivatives

Example 3. Find all functions g such that

$$g'(x) = \underset{①}{4\sin x} + \underset{②}{\frac{2x^5 - \sqrt{x}}{x}}$$

① $4\sin x \longrightarrow -4\cos x$

② $\dfrac{2x^5 - \sqrt{x}}{x} = 2x^4 - x^{-1/2}$

$\hookrightarrow \dfrac{2x^5}{5} - \dfrac{x^{1/2}}{1/2} = \dfrac{2}{5}x^5 - 2x^{1/2}$

③ Answer

$$g(x) = -4\cos x + \frac{2}{5}x^5 - 2x^{1/2} + C.$$

Initial Condition

Example 4. Find F if $F'(x) = x\sqrt{x}$ and $F(1) = 2$.

$$x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$$

$$\rightarrow \quad F(x) = \frac{x^{3/2+1}}{3/2+1} + C = \frac{x^{5/2}}{5/2} + C$$

$$\rightarrow \quad F(x) = \frac{2}{5} x^{5/2} + C$$

We have $F(1) = 2$

$$\Rightarrow \quad 2 = F(1) = \frac{2}{5} + C \quad \Rightarrow \quad C = \frac{8}{5}$$

$$\Rightarrow \quad \boxed{F(x) = \frac{2}{5} x^{5/2} + \frac{8}{5}}$$

Example 5. Find F if $F'(x) = \frac{1}{x^2}$ and $F(1) = 2$.

$$\frac{1}{x^2} = x^{-2} \quad \rightarrow \quad F(x) = \frac{x^{-2+1}}{-2+1} + C$$

$$\rightarrow \quad F(x) = -\frac{1}{x} + C$$

We have $F(1) = 2$

$$\Rightarrow \quad 2 = -\frac{1}{1} + C \quad \Rightarrow \quad C = 3$$

So, $\boxed{F(x) = -\frac{1}{x} + 3}$