MATH 241

CHAPTER 4

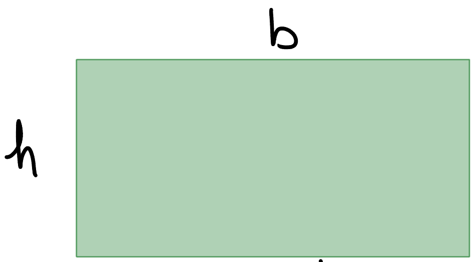
SECTION 4.1: AREAS AND DISTANCES

CONTENTS

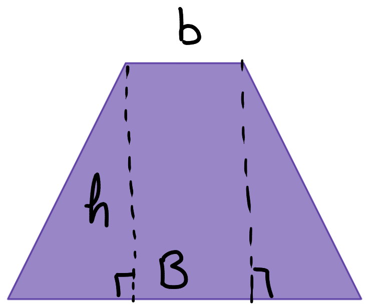
CREATED BY: PIERRE-OLIVIER PARISÉ

FALL 2022

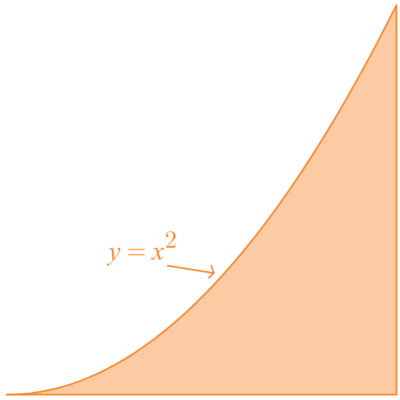
What is the area of the following shapes?



(a) Area = hb



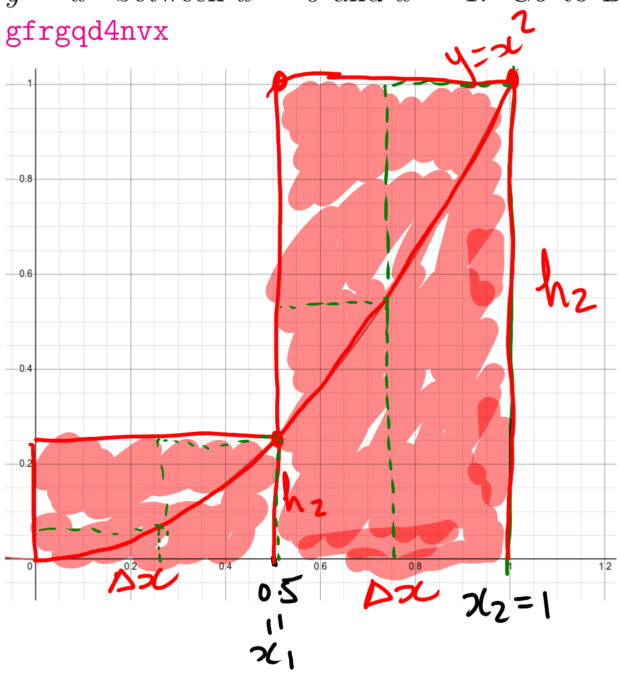
(b) Area = $\left(\frac{B+b}{2}\right)h$



(c) Area = $\frac{1}{3}$

Trick: Use simpler shapes, such as rectangles, to approximate the area.

Example 1. Using rectangles, approximate the area of the region S under the graph of $y = x^2$ between $x = 0$ and $x = 1$. Go to Desmos: https://www.desmos.com/calculator/gfrgqd4nvx



Step 1) $n=2$ squares

↳ Divide $[0,1]$ into 2 subintervals.

$$\Delta x = \frac{1-0}{2} = 0.5$$

$[0, 0.5]$ & $[0.5, 1]$

Step 2) Choose Rightend point

$x_1 = 0.5$, $x_2 = 0.5+0.5 = 1$

Step 3) Draw rectangles.

$h_1 = f(x_1) = 0.5^2 = 0.25$, $h_2 = f(x_2) = 1^2 = 1$

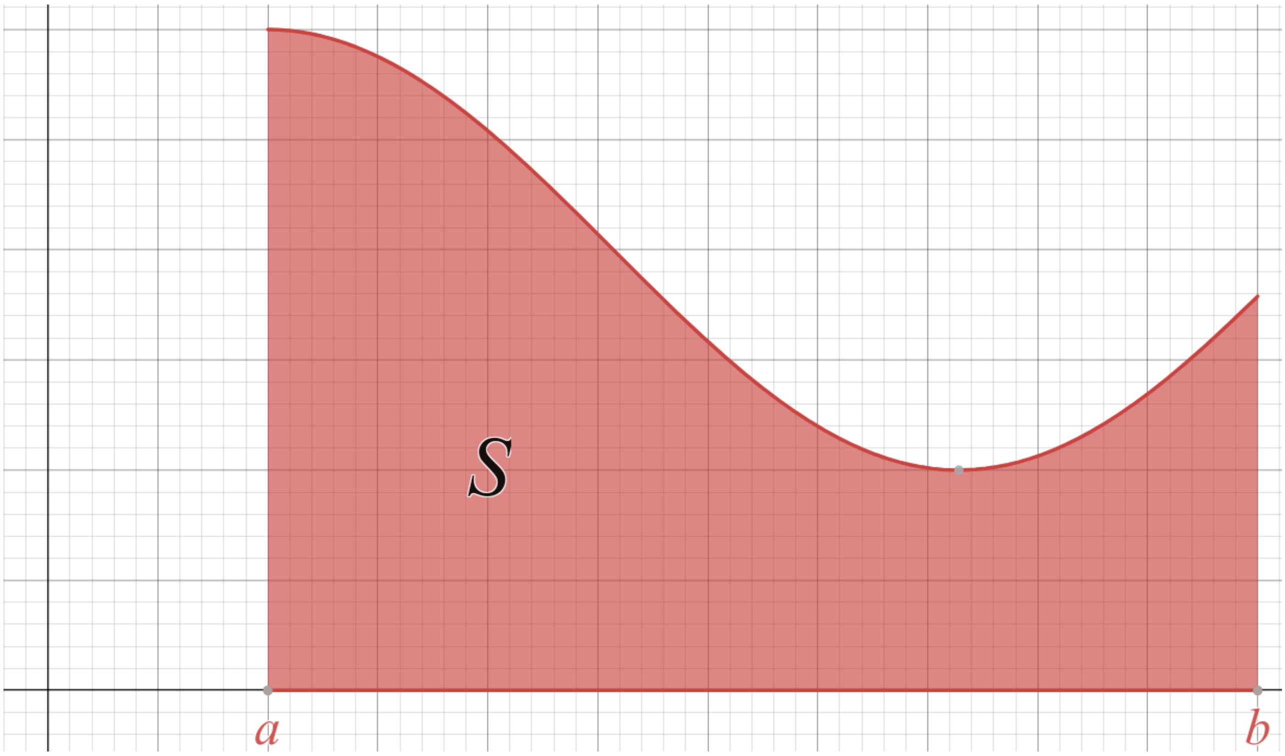
Step 4) Add area rectangles:

$$A \approx h_1 \cdot \Delta x + h_2 \cdot \Delta x = 0.25 \cdot 0.5 + 1 \cdot 0.5$$

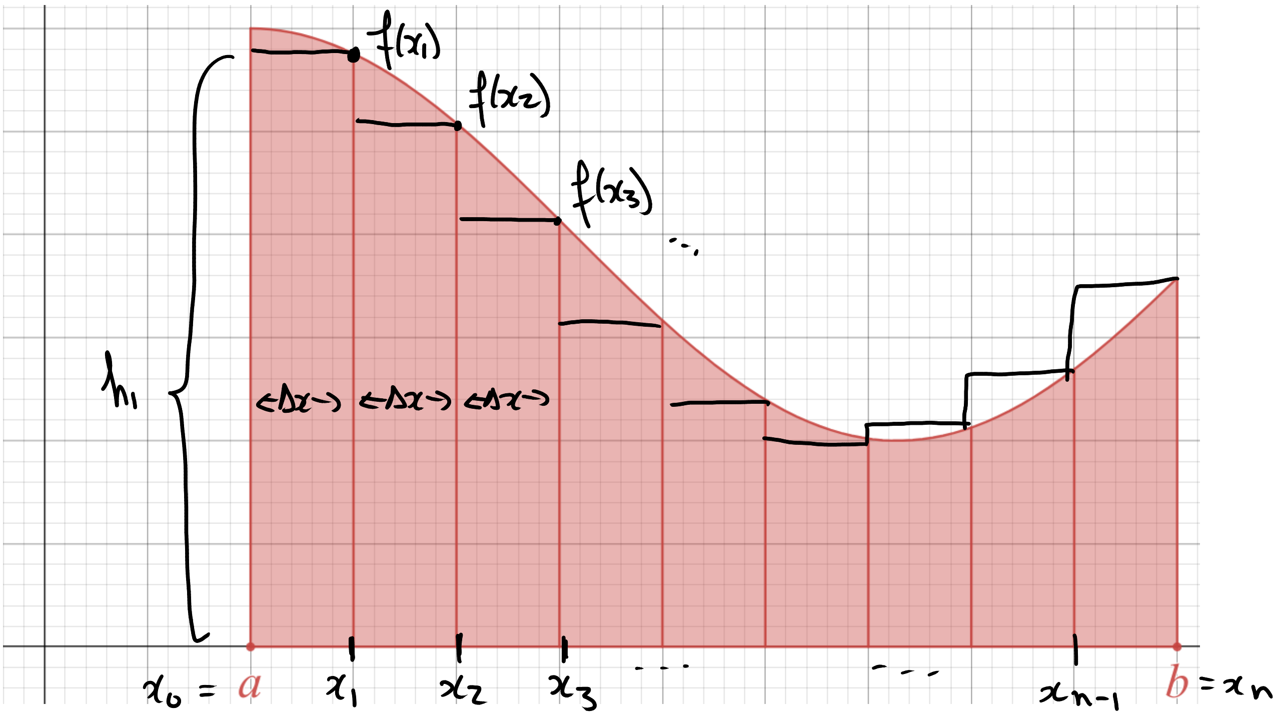
$$= 0.625$$

Divide and Conquer With the Right Endpoint Rule!

Suppose we want to compute the area of a region S bounded by the graph of some function $y = f(x)$.



STEP I Subdivide the region S into n strips of equal width $\Delta x = (b-a)/n$.



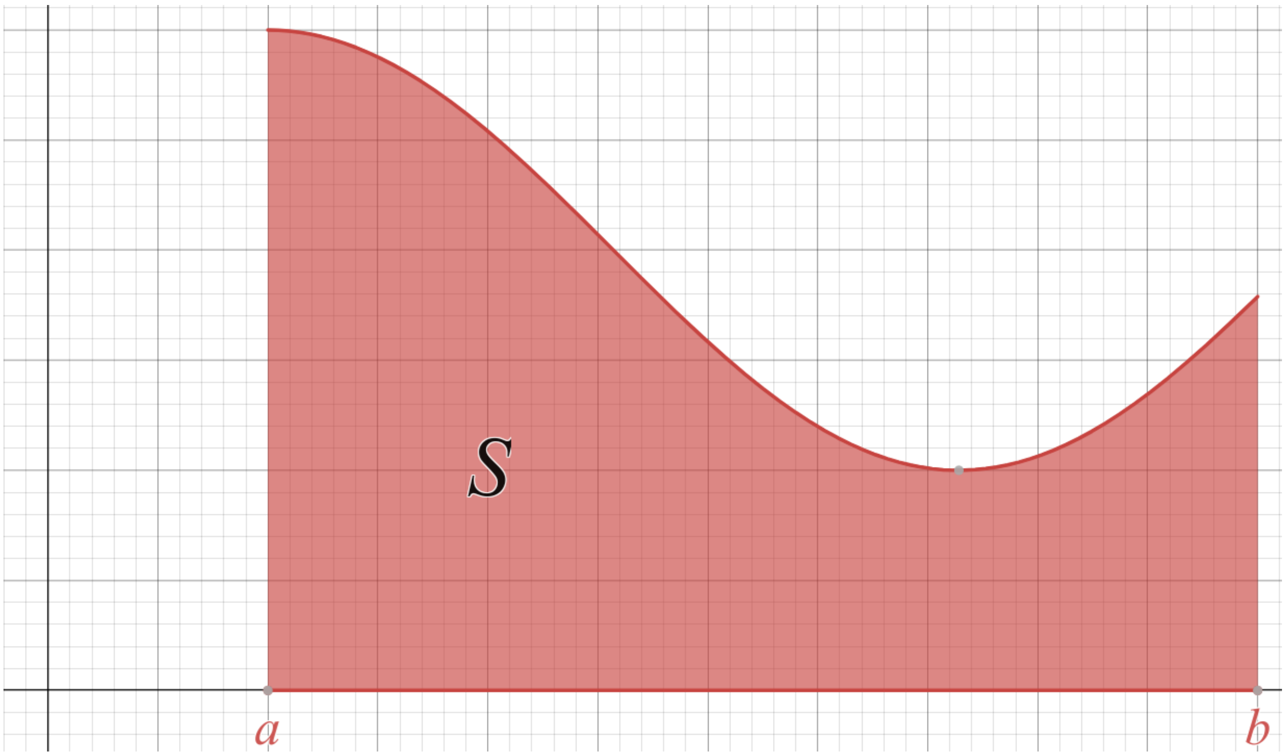
STEP II Choose the right-end point for all subintervals:
$x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\ldots$, $x_{n-1} = a + (n-1)\Delta x$, $x_n = b$.

STEP III Approximate by adding the area of each rectangle:

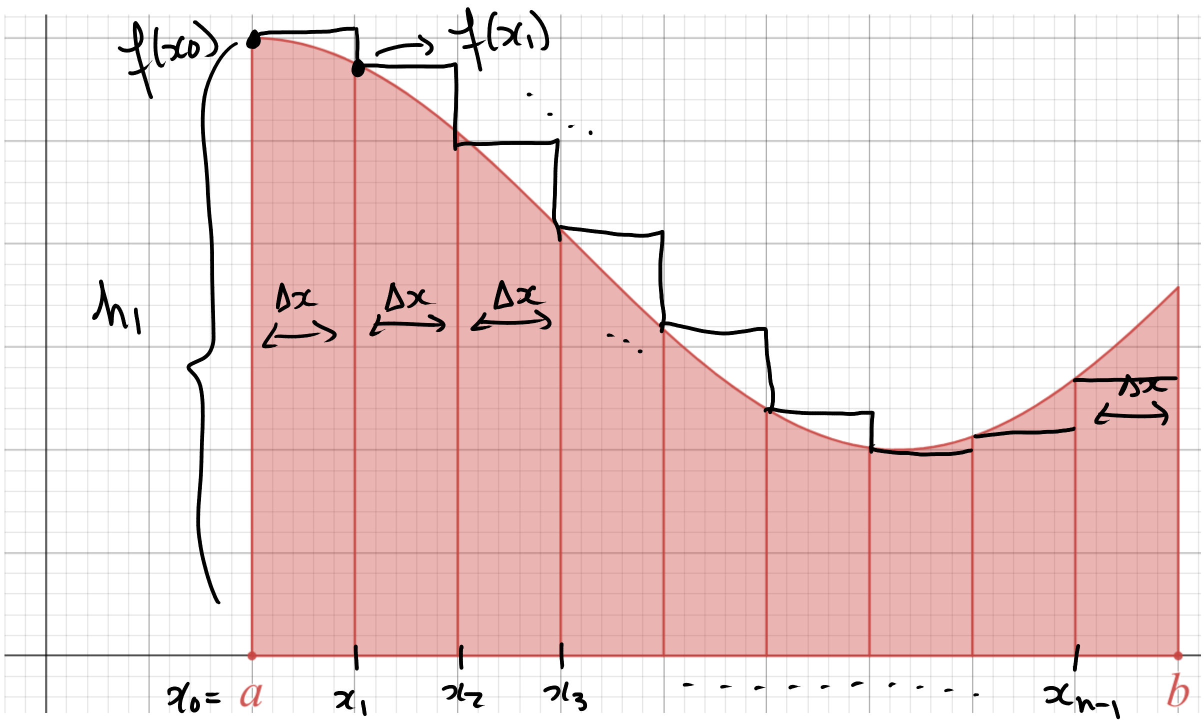
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Divide and Conquer With the Left Endpoint Rule!

Suppose we want to compute the area of a region S bounded by the graph of some function $y = f(x)$ from $x = a$ to $x = b$.



STEP I Subdivide the region S into n strips of equal width $\Delta x = (b - a)/n$.



STEP II Choose the left-end point for all subintervals:
$x_0 = a$, $x_1 = a + \Delta x$, …, $x_{n-2} = a + (n-2)\Delta x$, $x_{n-1} = a + (n-1)\Delta x$.

STEP III Approximate by adding the area of each rectangle:

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x.$$

Sigma Notation

We use the symbol $\sum$ to write a summation of numbers compactly:

ending index → n ← general term (notation : $a(i)$)

$$\sum_{i=k}^{n} a_i$$

index variable ← $i=k$ ← starting index

Example 2.

a) Expand $\sum_{i=1}^{7} i$.

b) Write $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$ with the Sigma notation.

c) Write $1+3+5+7+9+11+13$ with the Sigma notation.

a) general term: $a_i = i$ $[a(i) = i]$

start. index : 1

end. index : 7

$$\sum_{i=1}^{7} i = 1+2+3+4+5+6+7 = 28$$

b) $\sum_{i=1}^{7} \frac{1}{i}$ general term: $\frac{1}{i}$ start index: 1

end. index: 7

c) general term: $2i-1$ → $\sum_{i=1}^{7}(2i-1)$

↳ $\sum_{i=0}^{6}(2i+1)$

Useful Sum Formulas:

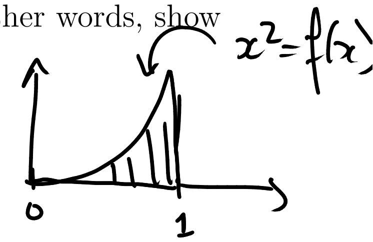
- $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$;
- $\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$;
- $\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

Taking the Limit!

Example 3. Show that the area of the region S in Example 1 is $1/3$. In other words, show that

$$\int_0^1 x^2\,dx = \text{Area}(S) = \lim_{n\to\infty} R_n = 1/3.$$



<u>Expression of R_n</u> $\quad a = 0 \qquad b = 1$

$$f(x) = x^2$$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_1 = 0 + 1\,\Delta x = \frac{1}{n}, \quad x_2 = 0 + 2\cdot\Delta x = \frac{2}{n}, \ldots$$

$$x_i = 0 + i\cdot\Delta x = \frac{i}{n}, \ldots, x_n = 0 + n\,\Delta x = 1$$

So,
$$R_n = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + \Delta x\, f(x_i) + \Delta x\, f(x_n)$$

$$= \sum_{i=1}^{n}\left(f(x_i)\cdot\frac{1}{n}\right)$$

$$\Rightarrow R_n = \sum_{i=1}^{n}\frac{i^2}{n^2}\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{i^2}{n^3}.$$

<u>Take limit</u>. (1) $R_n = \dfrac{1}{n^3}\displaystyle\sum_{i=1}^{n} i^2 = \dfrac{1}{n^3}\left(\dfrac{n(n+1)(2n+1)}{6}\right)$

(2) $\displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3} = \frac{2}{6} = \boxed{\frac{1}{3}}$

<u>General definition of Area:</u> The area of the region S lying under the graph of a function $y = f(x)$ from $x = a$ to $x = b$ is given by

- $\text{Area}(S) = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\Big(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\Big)$
- $\text{Area}(S) = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\Big(f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x\Big)$

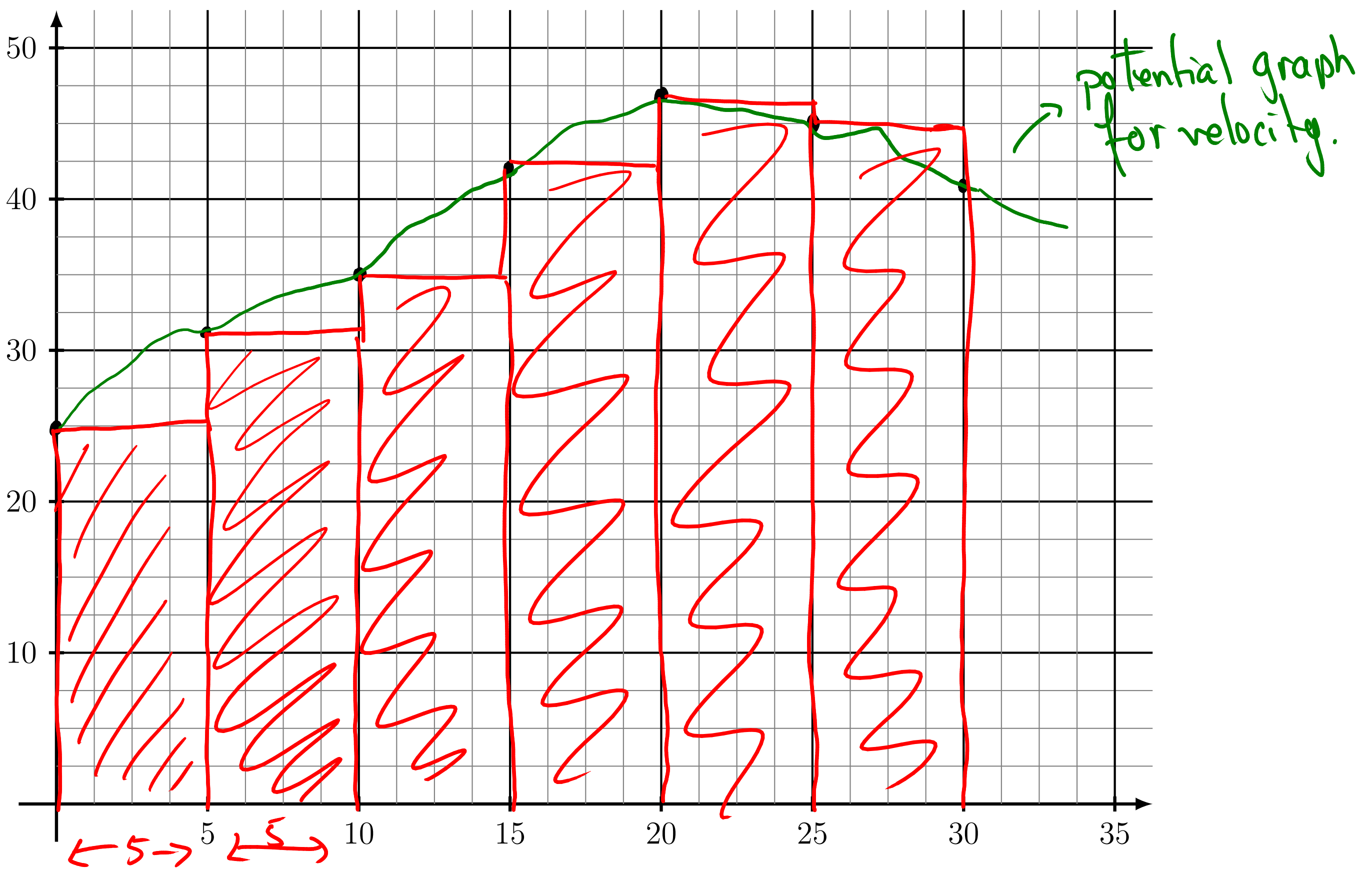
If an object move at constant velocity, then the distance between the start and finish line is easy to compute:

$$\text{DISTANCE} = \text{VELOCITY} \times \Delta\text{TIME} .$$

What do we do if the velocity is not constant?

EXAMPLE 4. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41



Left-hand point rule:

$\Delta x = 5,$

$x_0 = 0, \ x_1 = 5, \ x_2 = 10$

$x_3 = 15, x_4 = 20, x_5 = 25$

$$\text{Distance} \approx \sum_{i=0}^{5} \underbrace{f(x_i)}_{\text{from the table}} \Delta x$$

$$= 25 \cdot 5 + 31 \cdot 5 + 35 \cdot 5 + 43 \cdot 5$$
$$+ 47 \cdot 5 + 45 \cdot 5$$
$$= \boxed{1130 \text{ ft}}$$

Remark:

- The total distance is given by the area under the curve of the velocity function!