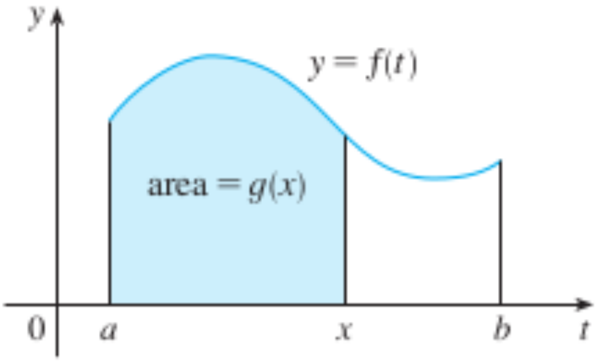
Chapter 4
Integrals

4.3 The Fundamental Theorem of Calculus

Area Function.



$$g(x) = \int_a^x f(t)\,dt$$

$$a \le x \le b$$

EXAMPLE 1 If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

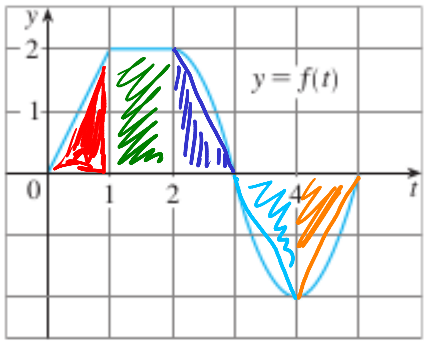


FIGURE 2

$\underline{g(0)}$ $\quad g(0) = \int_0^0 f(t)\,dt = 0$

$\underline{g(1)}$ $\quad g(1) = \int_0^1 f(t)\,dt = \text{Area}(\triangle) = 1$

$\underline{g(2)}$ $\quad g(2) = \int_0^2 f(t)\,dt = \text{Area}(\triangle) + \text{Area}(\square) = 3$

$\underline{g(3)}$ $\quad g(3) = \int_0^3 f(t)\,dt \approx \text{Area}(\triangle) + \text{Area}(\square) + \text{Area}(\triangle)$

$= 1 + 2 + 1 = 4$

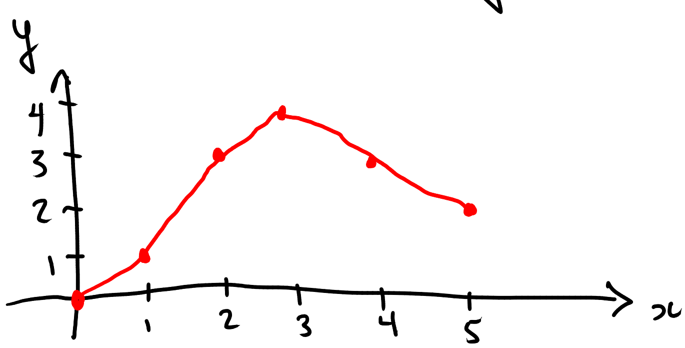
$\underline{g(4)}$ $\quad g(4) = \int_0^4 f(t)\,dt = \int_0^3 f(t)\,dt + \int_3^4 f(t)\,dt$

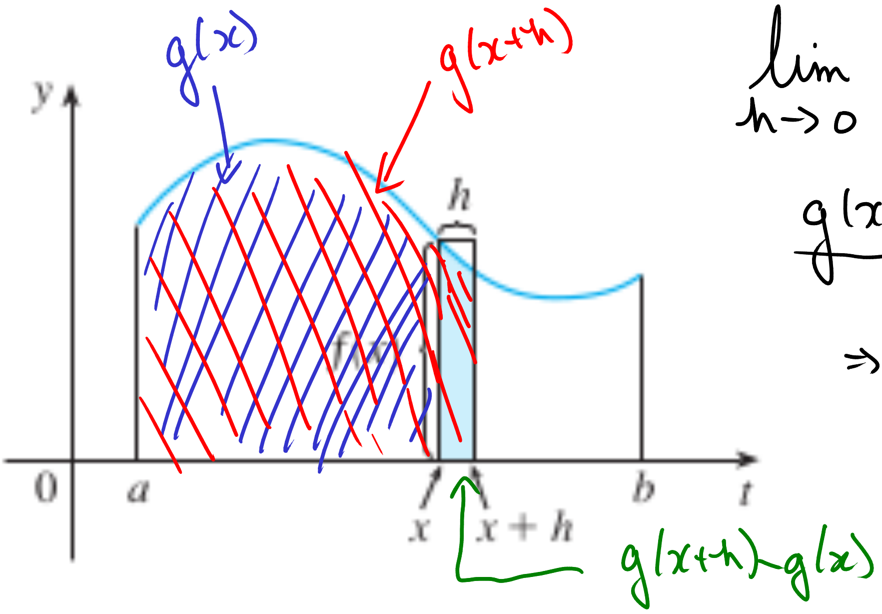
$= g(3) - \text{Area}(\triangledown) = 4 - 1 = 3$

$\underline{g(5)}$ $\quad g(5) = \int_0^5 f(t)\,dt = \int_0^4 f(t)\,dt + \int_4^5 f(t)\,dt$

$= g(4) - \underbrace{\text{Area}(\triangledown)}_{=1} = 2$

$\underline{\text{Graph:}}$





$$\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = g'(x).$$

$$\frac{g(x+h)-g(x)}{h} \approx \frac{h \cdot f(x)}{h}$$

$$\Rightarrow \frac{g(x+h)-g(x)}{h} \approx f(x)$$

$$\overset{h \to 0}{\Rightarrow} \quad g'(x) = f(x).$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$.

$$g'(x) = \sqrt{1+x^2}$$

① Identify integrand : $f(t) = \sqrt{1+t^2}$

② Apply FTC part I. : $g'(x) = f(x)$
$= \sqrt{1+x^2}$.

Example. Find $\frac{d}{dx}\Big(\underbrace{\int_1^{x^4} \sec(t)\,dt}_{g(x)}\Big)$.

$$\boxed{g'(x) \neq \sec(x)}$$

$$G(x) = \int_1^x \sec(t)\,dt \longrightarrow G'(x) = \underline{\sec(x)}$$

$$\hookrightarrow \underbrace{G(x^4)}_{\substack{\text{composition} \\ \text{of } G \text{ \& } x^4.}} = \int_1^{x^4} \sec(t)\,dt = g(x)$$

$$\Rightarrow \quad \frac{d}{dx}\big(g(x)\big) = \frac{d}{dx}\left[G(x^4)\right]$$

$$= G'(x^4) \cdot \frac{d}{dx}(x^4)$$

$$= \boxed{\sec(x^4) \cdot 4x^3}$$

Example. Find the derivative of the function $f(x) = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$

$$\int_b^a f(x)\, dx = -\int_a^b f(x)$$

Apply with $a=1$ & $b=\sin x$

$$\Rightarrow \int_{\sin x}^{1} \sqrt{1+t^2}\, dt = -\int_1^{\sin x} \sqrt{1+t^2}\, dt$$

So,

$$f'(x) = -\sqrt{1+(\sin x)^2} \cdot \frac{d}{dx}(\sin x)$$

$$= \boxed{-\cos(x)\sqrt{1+\sin^2 x}}$$

Second part of the Fundamental Theorem of Calculus.

Example. Compute the integral $\int_a^b x\,dx$ where a and b are two numbers such that a < b.

Formula

$$\int_a^b x^n\,dx = \frac{b^{n+1} - a^{n+1}}{n+1}$$

$$x^n \xrightarrow{\text{Anti-derivative}} \frac{x^{n+1}}{n+1} \Rightarrow \int_a^b x^n\,dx = F(b) - F(a)$$

$$\frac{x^{n+1}}{n+1} \downarrow F(x)$$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Example. Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

Notation:

$$x^3 \xrightarrow{\text{Anti-derivative}} \frac{x^4}{4} + \cancel{C}^{=0} = F(x)$$

$$F(x)\Big|_{-2}^{1}$$

$$\Rightarrow \int_{-2}^{1} x^3\,dx = F(1) - F(-2)$$

$$= \left(\frac{1}{4} + \cancel{C}\right) - \left(\frac{16}{4} + \cancel{C}\right) = \frac{-15}{4}$$

Example. Find the value of the integral $\int_0^1 (3x^2 - \cancel{\sin(\pi x)} + \cos(x))\, dx.$

$\rightarrow \int_0^1 3x^2 + \cos x \, dx$

$$\int_0^1 3x^2 + \cos x \, dx = \int_0^1 3x^2 \, dx + \int_0^1 \cos x \, dx$$

$$= 3 \int_0^1 x^2 \, dx + \int_0^1 \cos x \, dx$$

$$= 3 \left. \frac{x^3}{3} \right|_0^1 + \left. \sin x \right|_0^1$$

$$= 3\left(\frac{1^3}{3} - \frac{0^3}{3} \right) + (\sin(1) - \sin 0)$$

$$= \boxed{1 + \sin(1)}$$

EXAMPLE 8 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

The interval $[-1, 3]$ "contains" a vertical asymptote !.

The calculations are not legit !

Differentiation and Integration as Inverse Processes.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.