Chapter 4
Integrals

4.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integral.

Previously on Calc I:

Fondamental Theorem of Calculus $\displaystyle\int_a^b f(x)dx = F(b) - F(a).$

We introduce a notation for the antiderivatives:

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

Example.

a) $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C$.

b) $\displaystyle\int \cos x\,dx = \sin x + C$.

c) $\displaystyle\int \sec^2 x\,dx = \tan x + C$.

Table of Indefinite integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Remark: We adopt the convention that the general indefinite integral is valid on a given interval. Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$.

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\, dx$$

$$2x^5 - 2\tan x + c$$

$$= 10\int x^4\, dx - 2\int \sec^2 x\, dx$$

$$= 10\left(\frac{x^5}{5}\right) - 2\tan x + C$$

EXAMPLE 2 Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$.

$$F(x) = \frac{\sin x}{2\sin x} \cdot \cos x$$

$$= \frac{\cos x}{2}$$

$$\frac{\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta} = \cot(\theta)\ \csc(\theta)$$

$$\Rightarrow \int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \int \cot(\theta)\csc(\theta)\, d\theta$$

$$= \boxed{-\csc(\theta) + C}$$

EXAMPLE 4 Find $\int_0^{12} (x - 12\sin x)\,dx$.

$$\int_0^{12} x - 12\sin x\,dx = \int_0^{12} x\,dx - 12\int_0^{12} \sin x\,dx$$

$$= \left(\frac{x^2}{2} + 12\cos x\right)\Big|_0^{12}$$

$$= \frac{12^2}{2} + 12\cos(12) - 12\cos(0)$$

$$= \frac{144}{2} + 12\cos(12) - 12$$

$$= 72 - 12 + 12\cos(12) = \boxed{60 + 12\cos(12)}$$

EXAMPLE 5 Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt.$ $= I$

$$\frac{2t^2 + t^2\sqrt{t} - 1}{t^2} = \frac{2\cancel{t^2}}{\cancel{t^2}} + \frac{\cancel{t^2}\sqrt{t}}{\cancel{t^2}} - \frac{1}{t^2}$$

$$= 2 + \sqrt{t} - \frac{1}{t^2} = 2 + t^{1/2} - t^{-2}$$

$$\Rightarrow I = \int_1^9 2 + t^{1/2} - t^{-2}\,dt = \left(2t + \frac{2t^{3/2}}{3} + t^{-1}\right)\Big|_1^9$$

$$= 2\cdot 9 + \frac{2\cdot 9^{3/2}}{3} + \frac{1}{9} - \left(2 + \frac{2}{3} + 1\right) = \underbrace{\boxed{32\tfrac{4}{9}}}_{32 + \frac{4}{9}}$$

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

a) Displacement: $v(t) = s'(t)$ (s: position)

$$\text{displacement} = \int_a^b v(t)\,dt = s(b) - s(a)$$

b) Total distance traveled:

$$\text{tot. distance travelled} = \int_a^b |v(t)|\,dt.$$

c) Acceleration: net change in velocity if $a(t) = v'(t)$

$$\Rightarrow \int_a^b a(t)\,dt = v(b) - v(a)$$

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $\underset{a}{1} \leqslant t \leqslant \underset{b}{4}$.

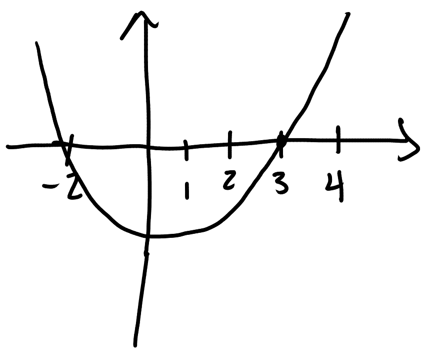
(b) Find the distance traveled during this time period.

(a) $$\text{displ.} = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_1^4 = \boxed{-4.5\,m}$$

(b) $$\text{tot. distance} = \int_1^4 |v(t)|\,dt = \int_1^4 |t^2 - t - 6|\,dt.$$

① $t^2 - t - 6 = (t-3)(t+2) = 0 \iff t = 3$ or $t = -2$



$$|t^2 - t - 6| = \begin{cases} -(t^2 - t - 6), & 1 \le t \le 3 \\ t^2 - t - 6, & 3 < t \le 4 \end{cases}$$

② $$\int_1^4 |t^2-t-6|\,dt = \int_1^3 -(t^2-t-6)\,dt + \int_3^4 t^2-t-6\,dt$$

$$= \int_1^3 -t^2+t+6\,dt + \int_3^4 t^2-t-6\,dt$$

$$= \left(-\frac{t^3}{3}+\frac{t^2}{2}+6t\right)\Bigg|_1^3 + \left(\frac{t^3}{3}-\frac{t^2}{2}-6t\right)\Bigg|_3^4$$

$$= \frac{61}{6} \approx \boxed{10.17\text{ m}}$$