Chapter 4
Integrals

4.5 The Substitution Rule

Example to start. Find the indefinite integral of $2x\sqrt{1+x^2}$, that is compute

$$\int 2x\sqrt{1+x^2}\,dx.$$

Answer: $\frac{2}{3}(1+x^2)^{3/2}$

Example, Take 2. Compute the indefinite integral

$$\int 2x\sqrt{1+x^2}\,dx.$$

1st: $\frac{d}{dx}(1+x^2) = 2x \Rightarrow d(1+x^2) = 2x\,dx$

2nd: $\boxed{u = 1+x^2} \rightarrow \frac{du}{dx} = \frac{d}{dx}(1+x^2) = 2x$

$\rightarrow du = 2x\,dx$

3rd: $\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx$

$= \int \sqrt{1+x^2}\,du$

$= \int \underbrace{\sqrt{u}}_{u^{1/2}}\,du = \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}(1+x^2)^{3/2} + C}$

Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\underbrace{g'(x)\,dx}_{} = \int f(u)\,du.$$

$2x\,dx$

Relation between du and dx:

$$u = g(x) \longrightarrow du = g'(x)\,dx$$

EXAMPLE 1 Find $\int x^3 \cos(x^4 + 2)\,dx$.

$$u = x^4 + 2 \Longrightarrow \frac{du}{dx} = 4x^3 \Rightarrow du = \boxed{4x^3\,dx}$$

$$\int x^3 \cos(x^4+2)\,dx = \int \frac{4x^3}{4}^{*} \cos(x^4+2)\,dx$$

$$= \frac{1}{4}\int 4x^3 \cos(x^4+2)\,dx$$

$$= \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{1}{4}\sin(u) + C$$

$$= \boxed{\frac{1}{4}\sin(x^4+2) + C}.$$

* Just multiplying by 4 will change the whole expr.
we have to rewrite 1 as $\frac{4}{4}$.

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\, dx$.

$u = 2x+1 \longrightarrow \frac{du}{dx} = 2 \longrightarrow du = 2dx$

$\longrightarrow \frac{du}{2} = dx$

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\, \frac{du}{2} = \frac{u^{3/2}}{3/2} + C$$

$$= \boxed{\frac{2}{3}(2x+1)^{3/2} + C.}$$

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1-4x^2}}\, dx$.

$u = 1-4x^2 \longrightarrow \frac{du}{dx} = -8x \longrightarrow du = -8x\,dx$

$\longrightarrow \frac{du}{-8} = x\,dx$

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx = \int \left[\frac{1}{\sqrt{u}}\right] \frac{du}{(-8)}$$

$$= -\frac{1}{8} \int u^{-1/2}\, du$$

$$= -\frac{1}{8} \frac{u^{1/2}}{1/2} + C$$

$$= -\frac{1}{4} u^{1/2} + C$$

$$= \boxed{-\frac{\sqrt{1-4x^2}}{4} + C}$$

EXAMPLE 5 Find $\int \sqrt{1+x^2}\, x^5\, dx$.

$$\sqrt{1+x^2}\quad x^5$$
$$= \sqrt{1+x^2}\quad (x^5)^{2/2}$$
$$= \sqrt{1+x^2}\quad (x^{10})^{1/2}$$
$$= \sqrt{x^{10}+x^{12}}$$

$$u = 1+x^2 \rightarrow du = 2x\,dx$$
$$\rightarrow \frac{du}{2} = x\,dx$$

$$\int \sqrt{1+x^2}\; x^5\,dx = \int \sqrt{u}\; x^4 \cdot x\,dx$$
$$= \int \sqrt{u}\; x^4\, \frac{du}{2}$$
$$= \frac{1}{2}\int \sqrt{u}\; x^4\,du$$

$$u = 1+x^2 \Rightarrow u-1 = x^2 \Rightarrow (u-1)^2 = x^4$$

$$\Rightarrow \text{integral} = \frac{1}{2}\int \sqrt{u}\,(u-1)^2\,du$$
$$= \frac{1}{2}\int u^{1/2}\left(u^2-2u+1\right)du$$
$$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\,du$$
$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - 2\,\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C$$
$$= \frac{u^{7/2}}{7} - \frac{2}{5}u^{5/2} + \frac{u^{3/2}}{3} + C$$
$$= \frac{(1+x^2)^{7/2}}{7} - \frac{2}{5}(1+x^2)^{5/2} + \frac{(1+x^2)^{3/2}}{3} + C.$$

$u = g(x)$

$$\Rightarrow \int_a^b f(\underbrace{g(x)}_{u}) \underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

EXAMPLE 7 Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

$u = 3 - 5x \longrightarrow du = -5\,dx \rightarrow \frac{du}{-5} = dx$

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx = \int_{3-5(1)}^{3-5(2)} \frac{1}{u^2}\,\frac{du}{-5}$$

$$= \frac{-1}{5}\int_{-2}^{-7} u^{-2}\,du$$

$$= \frac{-1}{5}\left(\frac{u^{-1}}{-1}\Bigg|_{-2}^{-7} \right)$$

$$= \frac{-1}{5}\left(\frac{(-7)^{-1} - (-2)^{-1}}{-1} \right)$$

$$= \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2} \right)$$

$$= \boxed{\frac{1}{14}}$$