Math 241

Chapter 5

Section 5.1: Area Between Curves

Contents

Created by: Pierre-Olivier Parisé
Fall 2022

Desmos: https://www.desmos.com/calculator/o7vvfgfwzy

Given two functions $f(x)$ and $g(x)$ such that

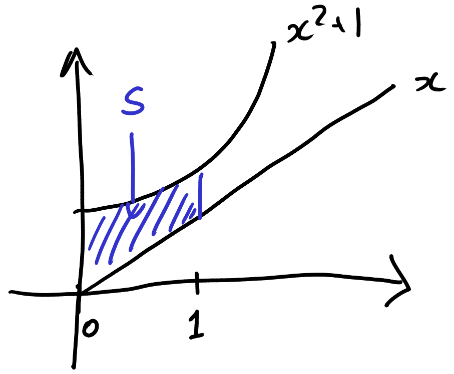
$$g(x) \leq f(x) \quad a \leq x \leq b,$$

the area of the region S enclosed by $f(x)$, $g(x)$, $x = a$ and $x = b$ is

$$\textsc{Area}\,(S) = \int_a^b f(x) - g(x)\, dx.$$

Example 1. Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

(1)



$f(x) = x^2+1$

$g(x) = x$

$a = 0$ & $b = 1$.

(2) $\text{Area}(S) = \int_0^1 x^2+1 - x \, dx$

$$= \left.\frac{x^3}{3} + x - \frac{x^2}{2}\right|_0^1$$

$$= \frac{1^3}{3} + 1 - \frac{1^2}{2} - \left(\frac{0^3}{3} + 0 - \frac{0^2}{2}\right)$$

$$= \frac{4}{3} - \frac{1}{2} - 0$$

$$= \frac{8-3}{6} = \boxed{\frac{5}{6}}$$

EXAMPLE 2. Find the area of the region enclosed by the functions $y = x^2$ and $y = x + 2$.

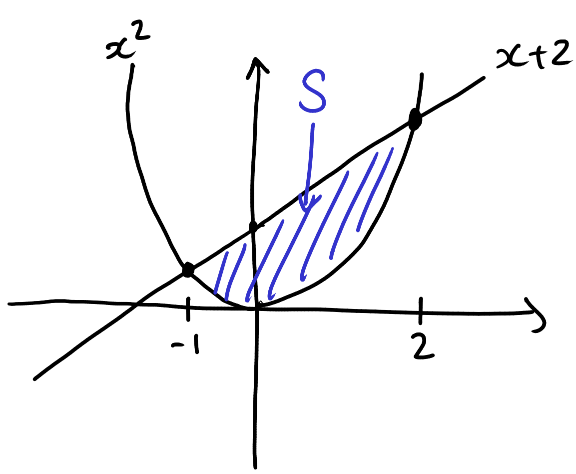
① Sketch & Intersects

$$x^2 = x + 2$$

$$\to x^2 - x - 2 = 0$$

$$\to (x-2)(x+1) = 0$$

$$\to x = 2 \text{ or } x = -1$$



② Integrate

$f(x) = x+2 \qquad g(x) = x^2 \qquad a = -1 \qquad b = 2$

$$\text{Area}(S) = \int_{-1}^{2} \underbrace{x+2}_{f(x)} - \underbrace{x^2}_{g(x)} \, dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3} \Big|_{-1}^{2}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= \boxed{\frac{9}{2}}$$

General Steps:

1. Draw a picture and find the points of intersection between the two curves.
2. Set up the definite integral and evaluate the definite integral.

Example 3. Find the area of the region enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. $\longrightarrow x = \frac{y^2}{2} - 3$

① Picture

(a) $y + 1 = x = \frac{y^2}{2} - 3$

$\longrightarrow \frac{y^2}{2} - y - 4 = 0$

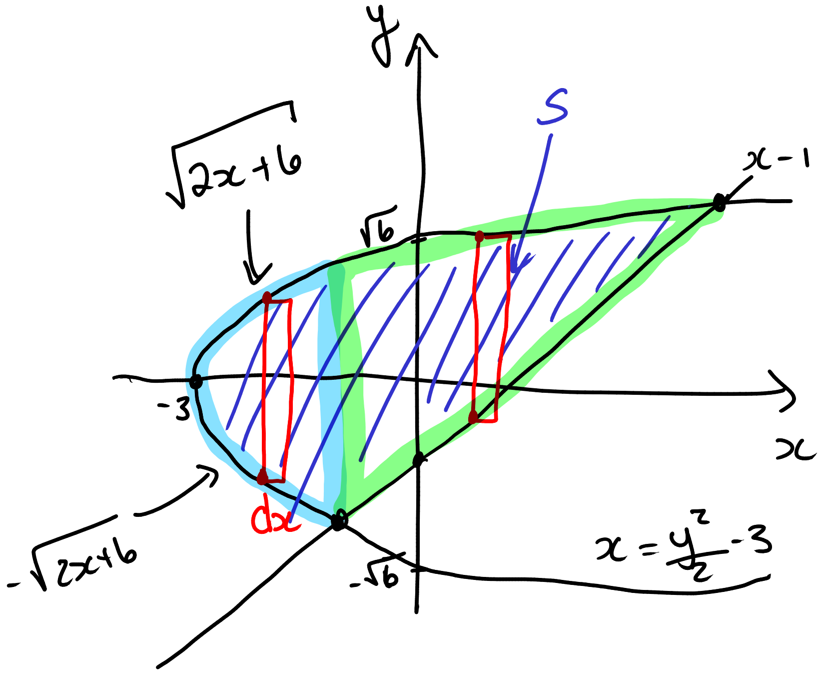
$\longrightarrow y^2 - 2y - 8 = 0$

$\longrightarrow (y-4)(y+2) = 0$

(b) $y = x - 1$

$\longrightarrow (x-1)^2 = x + 6$

$\longrightarrow x = 5$ or $x = -1$



$\frac{y^2}{2} - 3 = 0 \longrightarrow y^2 - 6 = 0$

$\longrightarrow y = \pm\sqrt{6}$

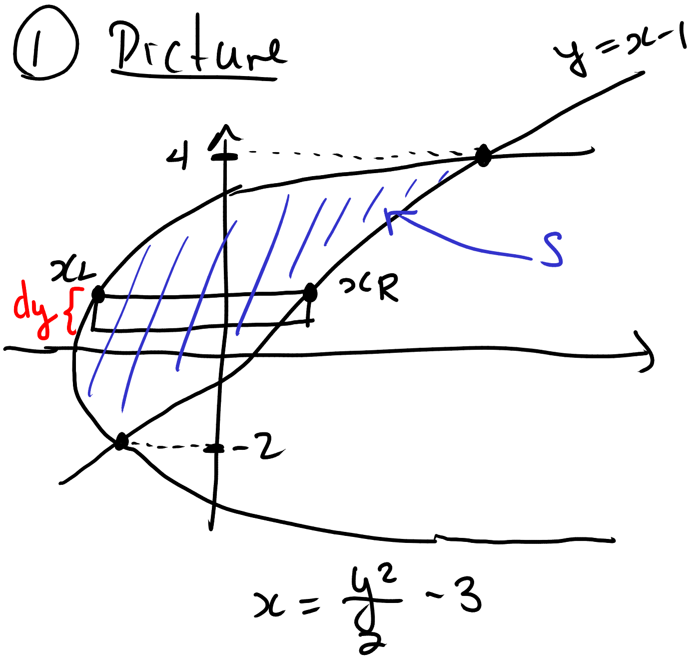
② Integrate

$f(x) = \sqrt{2x+6}$, $g(x) = \begin{cases} x - 1 \\ -\sqrt{2x+6} \end{cases}$ $\quad a = \quad\quad b =$

$$\text{Area}(S) = \boxed{\int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right)\, dx} + \boxed{\int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx}$$

EXAMPLE 4. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

① Picture



$x_R = y + 1$

$x_L = \frac{y^2}{2} - 3$

$\frac{y^2}{2} - 3 = y + 1$

$\to y = -2$ & $y = 4$

Note: "area" of $\underbrace{\boxed{}}_{x_R - x_L}\} dy$ $= \int_{-2}^{4} (x_R - x_L)\, dy$

gives you the whole area.

② Integrate

$$\text{Area}(S) = \int_{-2}^{4} x_R - x_L \, dy$$

$$= \int_{-2}^{4} (y+1) - \left(\frac{y^2}{2} - 3\right) dy$$

$$= \int_{-2}^{4} y + 4 - \frac{y^2}{2} \, dy$$

$$= \left(\frac{y^2}{2} + 4y - \frac{y^3}{6}\right) \Big|_{-2}^{4} = \boxed{18}$$