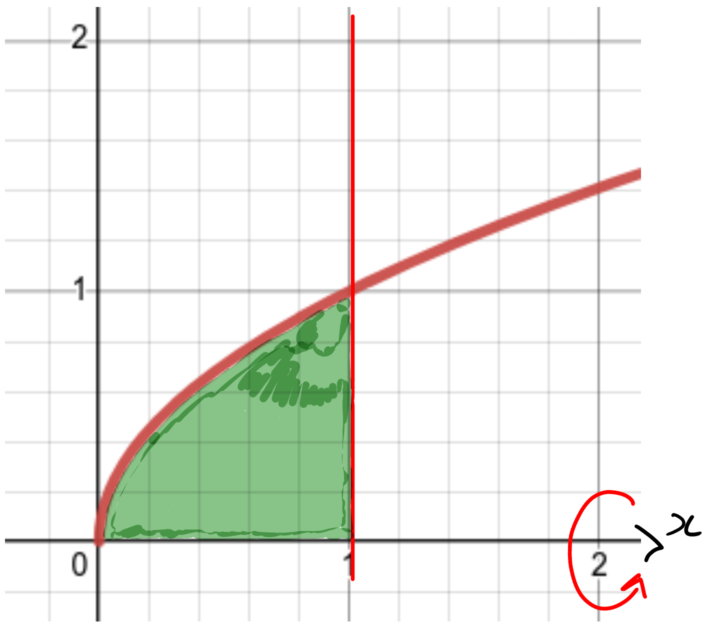
Chapter 5
Applications in integration

5.2 Volumes



$f(x) = \sqrt{x}$

- Consider the region enclosed by

$$x = 0 \;, \qquad x = 1 \;,$$

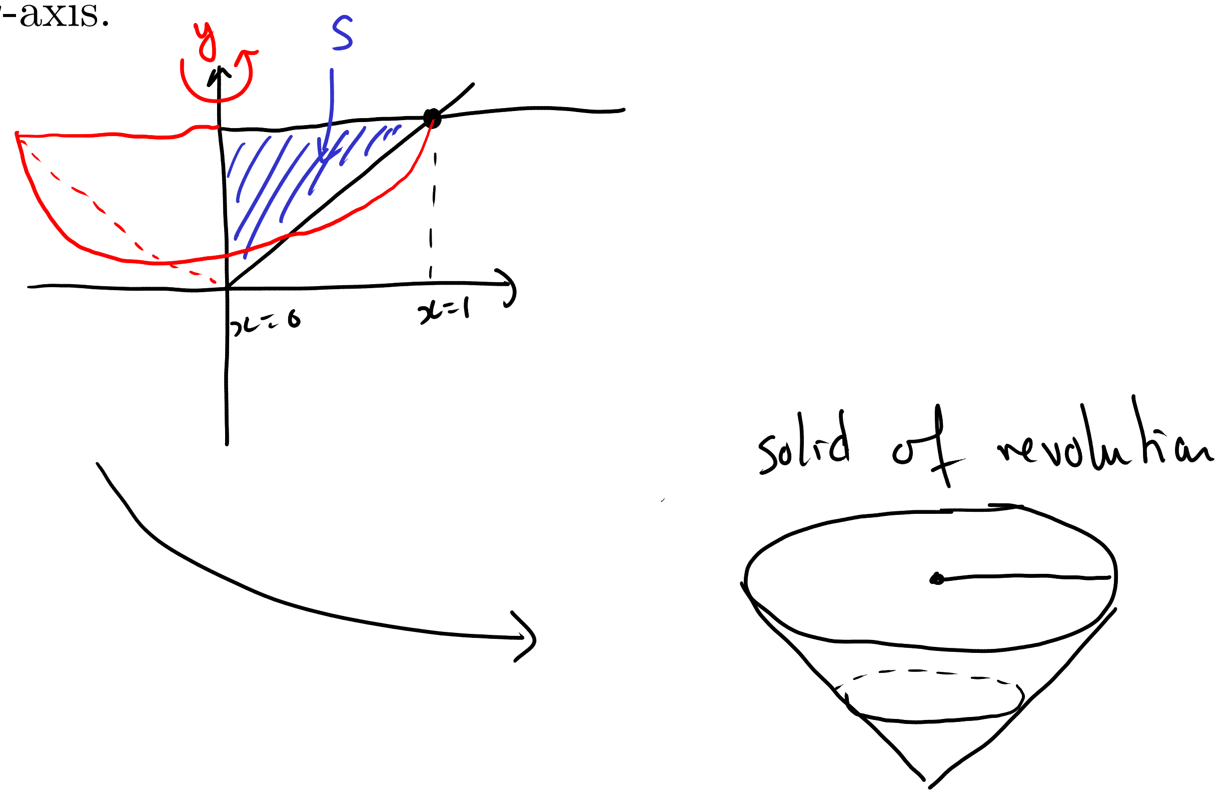
$$y = 0 \quad \text{and} \quad y = \sqrt{x}$$

- Rotate the region about one of the axis:
  - About x-axis

App: https://c3d.libretexts.org/CalcPlot3D/index.html#Volumes

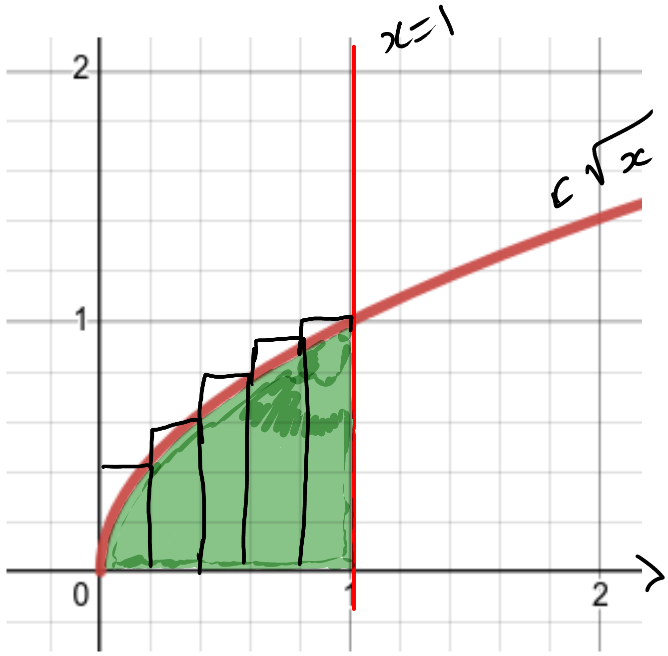
Example.

Rotate the region enclosed by $y = x, y = 1, x = 0$ about the y-axis.



VOLUME Of SOLIDS OF REVOLUTION.

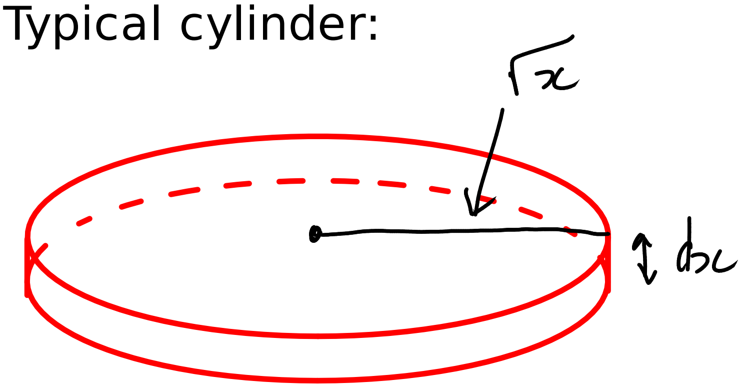
Rotation about the x-axis.



$f(x) = \sqrt{x}$

- Divide into subintervals.
- Create rectangles.
- Rotate those rectangles.
- Generate small cylinders

Typical cylinder:



- Radius: $\sqrt{x}$
- Heigth: dx

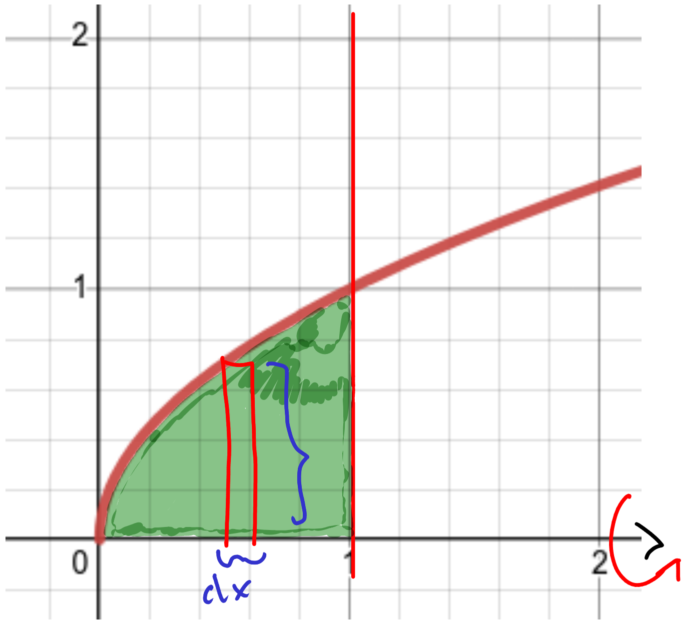
Volume of typical cylinder:

$$V = \pi (\text{radius})^2 \, h = \pi (\sqrt{x})^2 \, dx$$

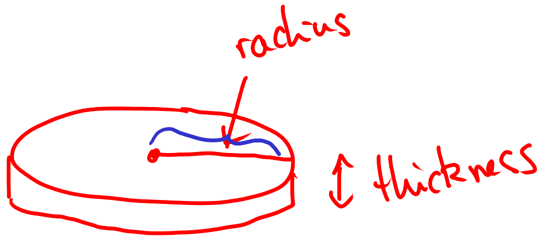
$$\boxed{\mathrm{Vol}(\mathrm{Solid}) = \int_a^b \pi (\mathrm{radius})^2 \, dx}$$

EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

SKETCH



$f(x) = \sqrt{x}$



radius $= \sqrt{x}$

thickness $= dx$

$a = 0$ to $b = 1$

$$\text{Vol (Solid)} = \int_0^1 \pi \, (\text{radius})^2 \, dx$$

$$= \int_0^1 \pi x \, dx$$

$$= \pi \frac{x^2}{2} \Big|_0^1$$

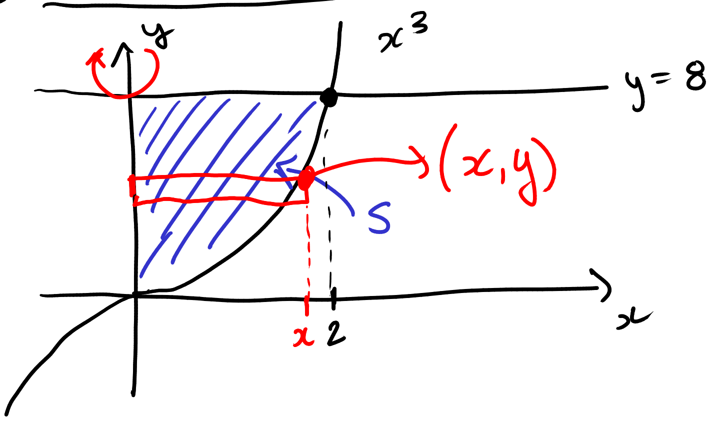
$$= \boxed{\frac{\pi}{2}}$$

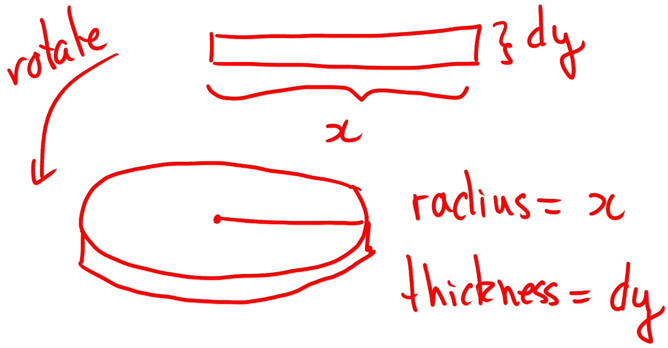
Rotation around the y-axis.

$$\mathrm{Vol}(\mathrm{Solid}) = \int_a^b \pi(\mathrm{radius})^2\, \underbrace{dy}_{\text{thickness}}$$

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

① Picture intersections $x^3 = 8 \Rightarrow x = 2$





② Volume

$a = 0$ and $b = 8$

$$\Rightarrow \mathrm{Vol}(\mathrm{Solid}) = \int_0^8 \pi\,(x)^2\, dy \qquad y = x^3 \Rightarrow \sqrt[3]{y} = x$$

$$= \int_0^8 \pi\,\left(y^{1/3}\right)^2\, dy$$

$$= \pi \int_0^8 y^{2/3}\, dy = \pi \left(\frac{y^{2/3+1}}{2/3+1} \right)\Bigg|_0^8$$

$$= \boxed{\frac{96\pi}{5}}$$

Cross-section as a washer.

Rotation about x-axis

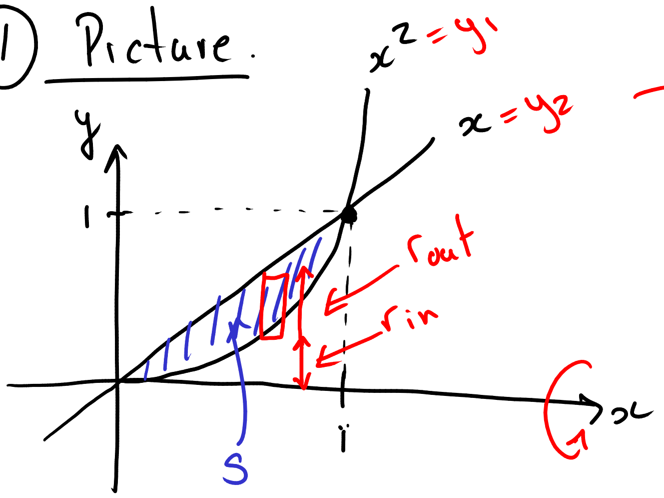
$$\text{Vol(Solid)} = \int_a^b \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\,dx$$

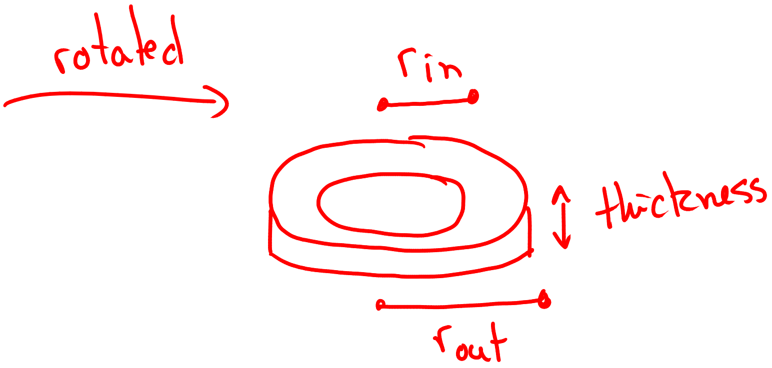
Rotation about y-axis

$$\text{Vol(Solid)} = \int_a^b \pi(r_{\text{out}}^2 - r_{\text{in}}^2)\,dy$$

EXAMPLE 4 The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

① Picture.





$r_{in} = y_1 = x^2$

$r_{out} = y_2 = x$

thickness $= dx$

$a = 0$, $b = 1$

② Volume

$$\text{Vol(Solid)} = \int_0^1 \left(\pi r_{out}^2 - \pi r_{in}^2\right) dx$$

$$= \int_0^1 \pi x^2 - \pi x^4 \, dx = \boxed{\frac{2\pi}{15}}$$