Chapter 5
Applications in integration

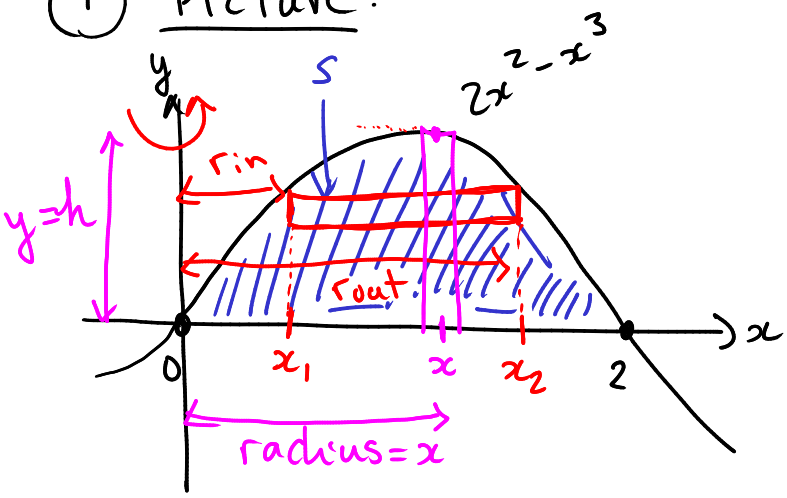
5.3 Volumes by Cylindrical Shells

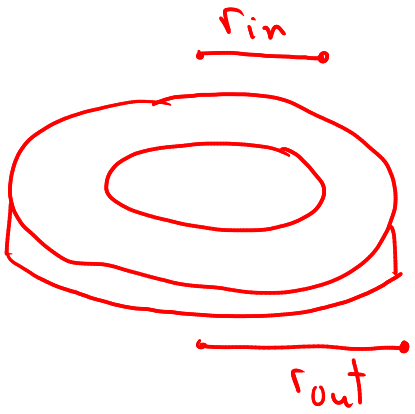
Illustrative Example. (Rotation about the y-axis)

Example 1.

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

① Picture.





$r_{in} = x_1$

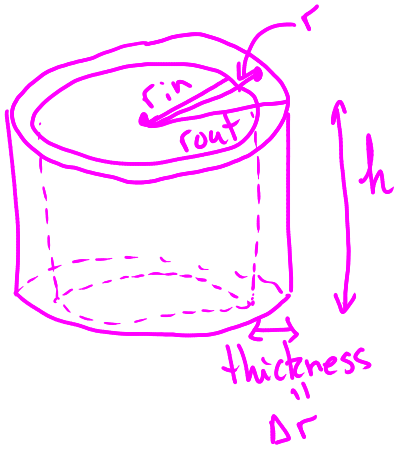
$r_{out} = x_2$

$$\text{Vol(Solid)} = \int_a^b \pi r_{out}^2 - \pi r_{in}^2 \, dy$$

$$= \int_a^b \pi x_2^2 - \pi x_1^2 \, dy$$

x_1 & x_2 are REALLY hard to find.

other rectangle



$$\pi r_{out}^2 h - \pi r_{in}^2 h$$

$$= \pi h (r_{out}^2 - r_{in}^2)$$

$$= \pi h (r_{out} - r_{in}) \left(\frac{r_{out} + r_{in}}{2}\right) \cdot 2$$

$$= \pi h \, \Delta r \cdot 2r$$

$$= 2\pi r h \cdot \Delta r$$

in our case: $h = y \Rightarrow$ Vol. $= 2\pi x \cdot y \cdot dx$

$r = x$

$\Delta r = dx$

radius, thickness

$$\text{Vol(Solid)} = \int_a^b 2\pi x y \, dx$$

height

(2) Volume.

$a = 0$
$b = 2$ $\longrightarrow$ $\text{Vol}(\text{Solid}) = \int_0^2 2\pi x y \, dx$

$$= \int_0^2 2\pi x (2x^2 - x^3) \, dx \quad (y = 2x^2 - x^3)$$

$$= 2\pi \int_0^2 2x^3 - x^4 \, dx$$

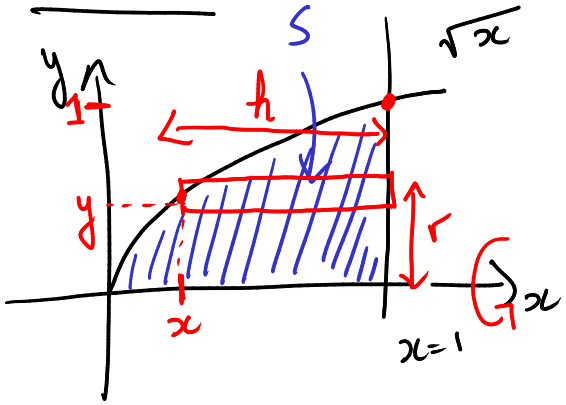
$$= \boxed{\frac{16\pi}{5}}$$

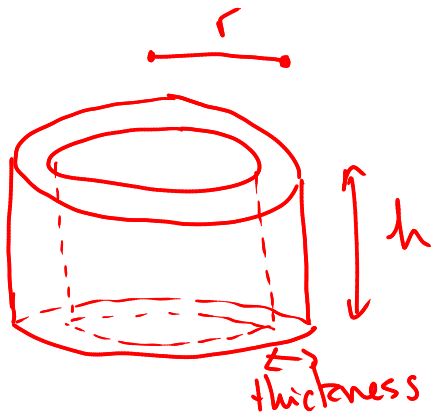
Example 3.

Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

$$\text{Vol(Solid)} = \int_a^b 2\pi (\text{height})(\text{radius})\, dy$$

① Picture





$r = y$

$h = 1 - x$

thickness $= dy$

$a = 0$

$b = 1$

② Volume

$$\text{Vol(Solid)} = \int_0^1 2\pi (1-x) \cdot y \, dy \qquad y = \sqrt{x} \to y^2 = x$$

$$= 2\pi \int_0^1 (1-y^2)\, y \, dy$$

$$= 2\pi \int_0^1 y - y^3 \, dy$$

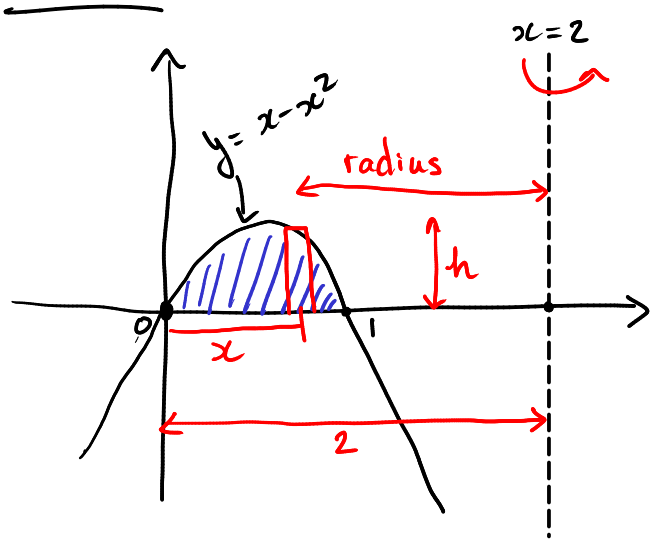
$$= \boxed{\frac{\pi}{2}}$$

Rotation about another axis. (Might be on exam)

Example 4.

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Picture.



$y = 0 \iff x - x^2 = 0 \iff x = 0, 1$

thickness = dx

radius = $2 - x$

height = $y = x - x^2$

$a = 0$

$b = 1$

Integrate

$$\text{Vol(Solid)} = \int_0^1 2\pi \,(\text{radius})(\text{height})\, dx$$

$$= 2\pi \int_0^1 (2-x)(x-x^2)\, dx$$

$$= 2\pi \int_0^1 2x - x^2 - 2x^2 + x^3 \, dx$$

$$= 2\pi \int_0^1 2x - 3x^2 + x^3 \, dx$$

$$= 2\pi \left(x^2 - x^3 + \frac{x^4}{4} \right)\Big|_0^1$$

$$= 2\pi \left[\left(1 - 1 + \frac{1}{4} \right) - 0 \right] = \boxed{\frac{\pi}{2}}$$