CHAPTER 1 CONCEPT CHECK ANSWERS

1. (a) What is a function? What are its domain and range?

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. The domain is the set D and the range is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) What is the graph of a function?

The graph of a function f consists of all points (x, y) such that $y = f(x)$ and x is in the domain of f.

(c) How can you tell whether a given curve is the graph of a function?

Use the Vertical Line Test: a curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

A function can be represented verbally, numerically, visually, or algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between the area A and side length s of a square: $A = s^2$ (an explicit formula)

3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.

A function f is even if it satisfies $f(-x) = f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the y-axis, then f is even. Examples are $f(x) = x^2$, $f(x) = \cos x$, $f(x) = |x|$.

(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

A function f is odd if it satisfies $f(-x) = -f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the origin, then f is odd. Examples are $f(x) = x^3$, $f(x) = \sin x$, $f(x) = 1/x$.

4. What is an increasing function?

A function f is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. What is a mathematical model?

A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon. (See the discussion on pages 23–24.)

6. Give an example of each type of function.

(a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^n$

(c) Exponential function: $f(x) = 2^x$, $f(x) = b^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

(e) Polynomial of degree 5: $f(x) = x^5 + 2x^4 - 3x^2 + 7$

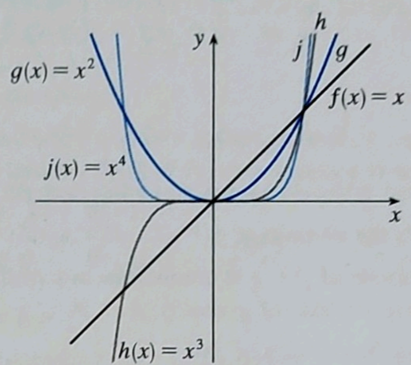
(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$

where $P(x)$ and $Q(x)$ are polynomials

7. Sketch by hand, on the same axes, the graphs of the following functions.

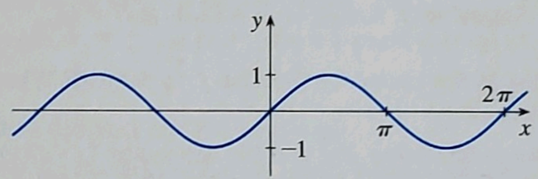
(a) $f(x) = x$ **(b)** $g(x) = x^2$

(c) $h(x) = x^3$ **(d)** $j(x) = x^4$

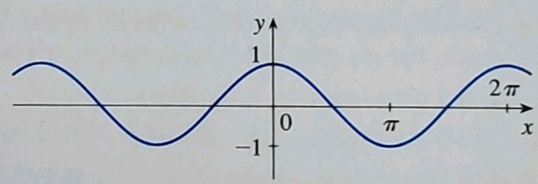


8. Draw, by hand, a rough sketch of the graph of each function.

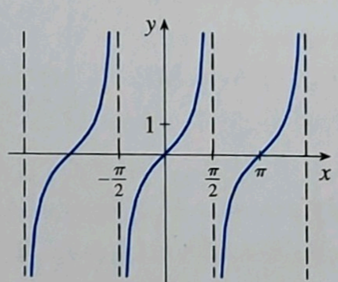
(a) $y = \sin x$



(b) $y = \cos x$

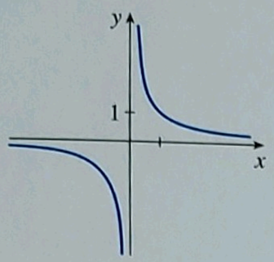


(c) $y = \tan x$

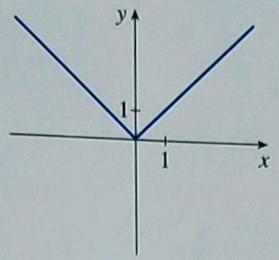


(continued)

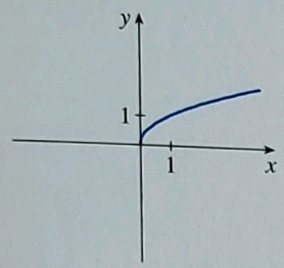
(d) $y = 1/x$



(e) $y = |x|$



(f) $y = \sqrt{x}$



9. Suppose that f has domain A and g has domain B.

(a) What is the domain of $f + g$?

The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) What is the domain of fg?

The domain of fg is also $A \cap B$.

(c) What is the domain of f/g?

The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. How is the composite function $f \circ g$ defined? What is its domain?

The composition of f and g is defined by $(f \circ g)(x) = f(g(x))$. The domain is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.

(a) Shift 2 units upward: $y = f(x) + 2$

(b) Shift 2 units downward: $y = f(x) - 2$

(c) Shift 2 units to the right: $y = f(x - 2)$

(d) Shift 2 units to the left: $y = f(x + 2)$

(e) Reflect about the x-axis: $y = -f(x)$

(f) Reflect about the y-axis: $y = f(-x)$

(g) Stretch vertically by a factor of 2: $y = 2f(x)$

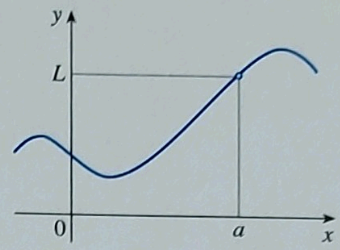
(h) Shrink vertically by a factor of 2: $y = \frac{1}{2}f(x)$

(i) Stretch horizontally by a factor of 2: $y = f\left(\frac{1}{2}x\right)$

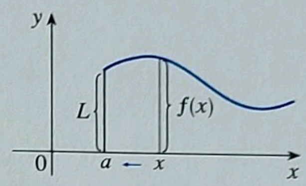
(j) Shrink horizontally by a factor of 2 $y = f(2x)$

12. Explain what each of the following means and illustrate with a sketch.

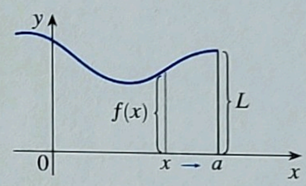
(a) $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a (but $x \neq a$).



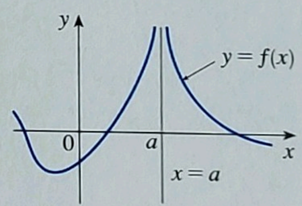
(b) $\lim_{x \to a^+} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values greater than a.



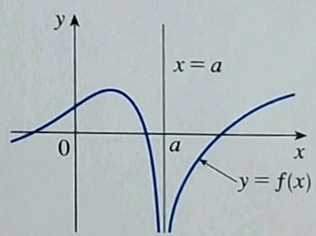
(c) $\lim_{x \to a^-} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values less than a.



(d) $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (but not equal to a).

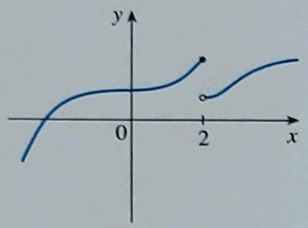


(e) $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a (but not equal to a).

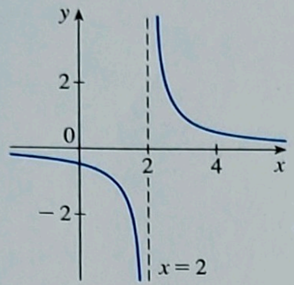


13. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

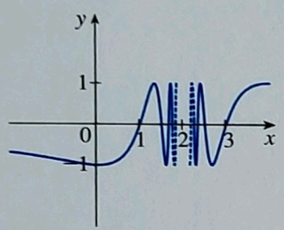
In general, the limit of a function fails to exist when the function values do not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left and right limits are not equal.



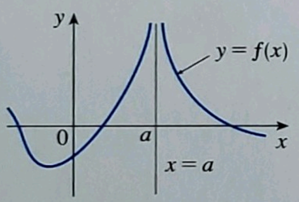
There is an infinite discontinuity.

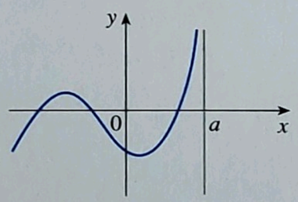


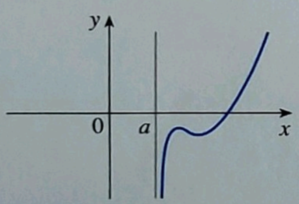
The function values oscillate between 1 and -1 infinitely often.

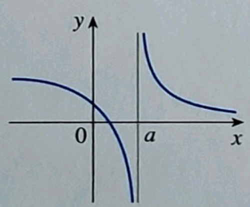
14. What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

It means that the limit of $f(x)$ as x approaches a from one or both sides is positive or negative infinity.









15. State the following Limit Laws.

(a) Sum Law

The limit of a sum is the sum of the limits:
$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(b) Difference Law

The limit of a difference is the difference of the limits:
$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(c) Constant Multiple Law

The limit of a constant times a function is the constant times the limit of the function: $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

(d) Product Law

The limit of a product is the product of the limits:
$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(e) Quotient Law

The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not 0:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$

(f) Power Law

The limit of a power is the power of the limit:
$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ (for n a positive integer)

(g) Root Law

The limit of a root is the root of the limit:
$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ (for n a positive integer)

16. What does the Squeeze Theorem say?

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$. In other words, if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

17. (a) What does it mean for f to be continuous at a?

A function f is continuous at a number a if the value of the function at $x = a$ is the same as the limit when x approaches a; that is, $\lim_{x \to a} f(x) = f(a)$.

(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

A function f is continuous on the interval $(-\infty, \infty)$ if it is continuous at every real number a.

The graph of such a function has no hole or break in it.

(continued)

18. **(a)** Give examples of functions that are continuous on $[-1, 1]$.

$f(x) = x^3 - x$, $g(x) = \sqrt{x + 2}$, $y = \sin x$, $y = \tan x$, $y = 1/(x - 3)$, and $h(x) = |x|$ are all continuous on $[-1, 1]$.

(b) Give an example of a function that is not continuous on $[0, 1]$.

$$f(x) = \frac{1}{x - \frac{1}{2}} \qquad \left[f(x) \text{ is not defined at } x = \tfrac{1}{2}\right]$$

19. What does the Intermediate Value Theorem say?

If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$ $[f(a) \neq f(b)]$, Then there exists a number c in (a, b) such that $f(c) = N$. In other words, a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.