CHAPTER 4 CONCEPT CHECK ANSWERS

1. (a) Write an expression for a Riemann sum of a function f on an interval $[a, b]$. Explain the meaning of the notation that you use.

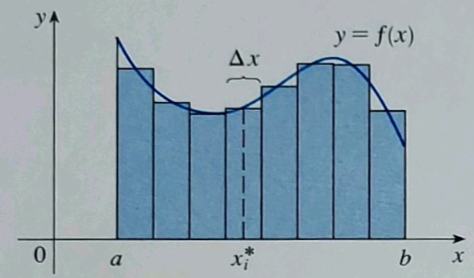
If f is defined for $a \leqslant x \leqslant b$ and we divide the interval $[a, b]$ into n subintervals of equal width Δx, then a Riemann sum of f is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is a point in the ith subinterval.

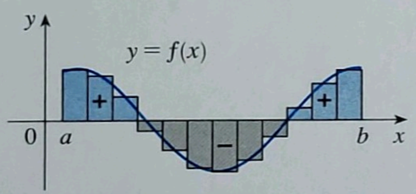
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f is positive, then a Riemann sum can be interpreted as the sum of areas of approximating rectangles, as shown in the figure.



(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f takes on both positive and negative values then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles minus the areas of the gray rectangles).



2. (a) Write the definition of the definite integral of a continuous function from a to b.

If f is a continuous function on the interval $[a, b]$, then we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

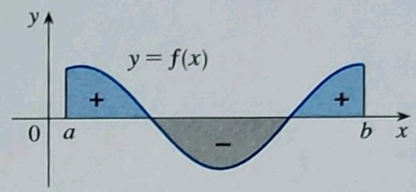
where x_i^* is any sample point in the ith subinterval $[x_{i-1}, x_i]$.

(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?

If f is positive, then $\int_a^b f(x)\,dx$ can be interpreted as the area under the graph of $y = f(x)$ and above the x-axis for $a \leqslant x \leqslant b$.

(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

In this case $\int_a^b f(x)\,dx$ can be interpreted as a "net area," that is, the area of the region above the x-axis and below the graph of f (labeled "+" in the figure) minus the area of the region below the x-axis and above the graph of f (labeled "−").



3. State the Midpoint Rule.

If f is a continuous function on the interval $[a, b]$ and we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$, then

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

where $\bar{x}_i$ = midpoint of $[x_{i-1}, x_i] = \frac{1}{2}(x_{i-1} + x_i)$.

4. State both parts of the Fundamental Theorem of Calculus.

Suppose f is continuous on $[a, b]$.

Part 1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

Part 2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

5. (a) State the Net Change Theorem.

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

$\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

(continued)

CONCEPT CHECK ANSWERS *(continued)*

6. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.

(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?

$\int_{60}^{120} v(t)\,dt$ represents the net change in position (the displacement) of the particle from $t = 60$ s to $t = 120$ s, in other words, in the second minute.

(b) What is the meaning of $\int_{60}^{120} |\,v(t)\,|\,dt$?

$\int_{60}^{120} |\,v(t)\,|\,dt$ represents the total distance traveled by the particle in the second minute.

(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

$\int_{60}^{120} a(t)\,dt$ represents the change in velocity of the particle in the second minute.

7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.

The indefinite integral $\int f(x)\,dx$ is another name for an antiderivative of f, so $\int f(x)\,dx = F(x)$ means that $F'(x) = f(x)$.

(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

The connection is given by Part 2 of the Fundamental Theorem:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

if f is continuous on $[a, b]$.

8. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

Part 1 of the Fundamental Theorem of Calculus can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f.

Since $F'(x) = f(x)$, Part 2 of the theorem (or, equivalently, the Net Change Theorem) states that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function, but in the form $F(b) - F(a)$.

Also, the indefinite integral $\int f(x)\,dx$ represents an antiderivative of f, so

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

9. State the Substitution Rule. In practice, how do you use it?

If $u = g(x)$ is a differentiable function and f is continuous on the range of g, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

In practice, we make the substitutions $u = g(x)$ and $du = g'(x)\,dx$ in the integrand in order to make the integral simpler to evaluate.