Math 241

Chapter 4

Section 4.2: Definite Integral

Contents

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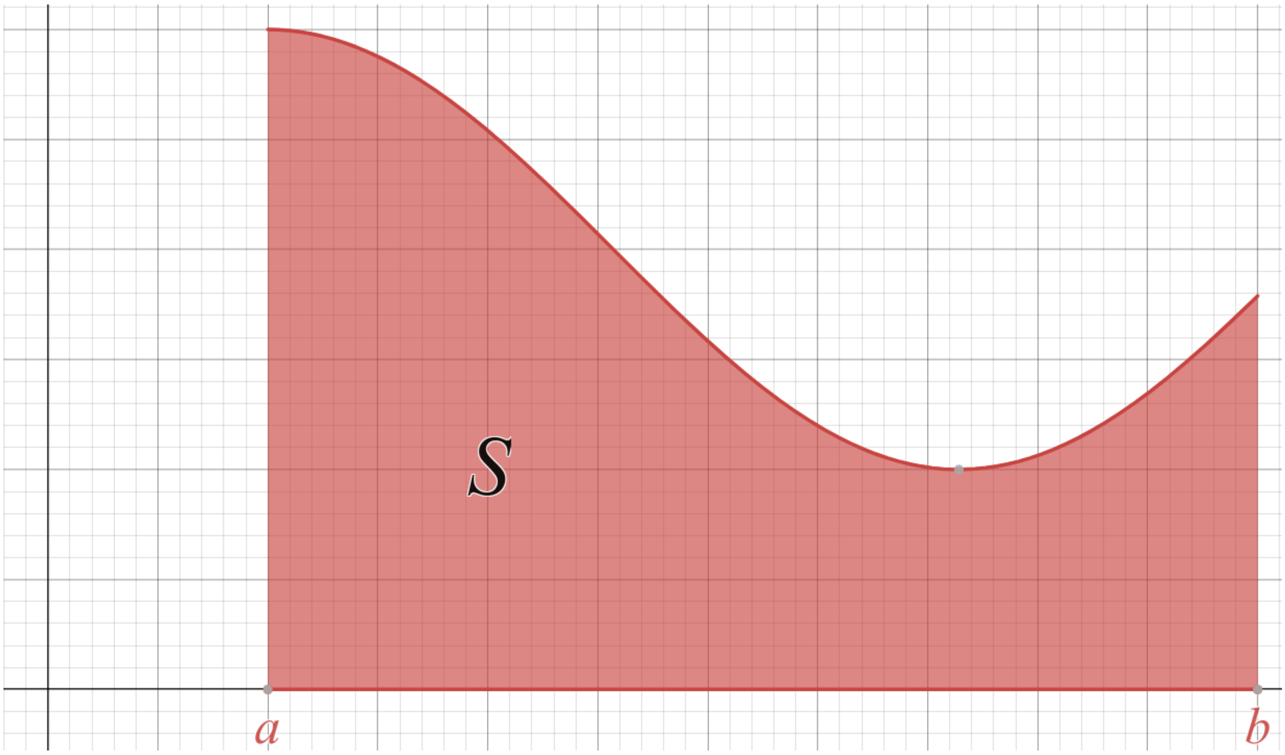
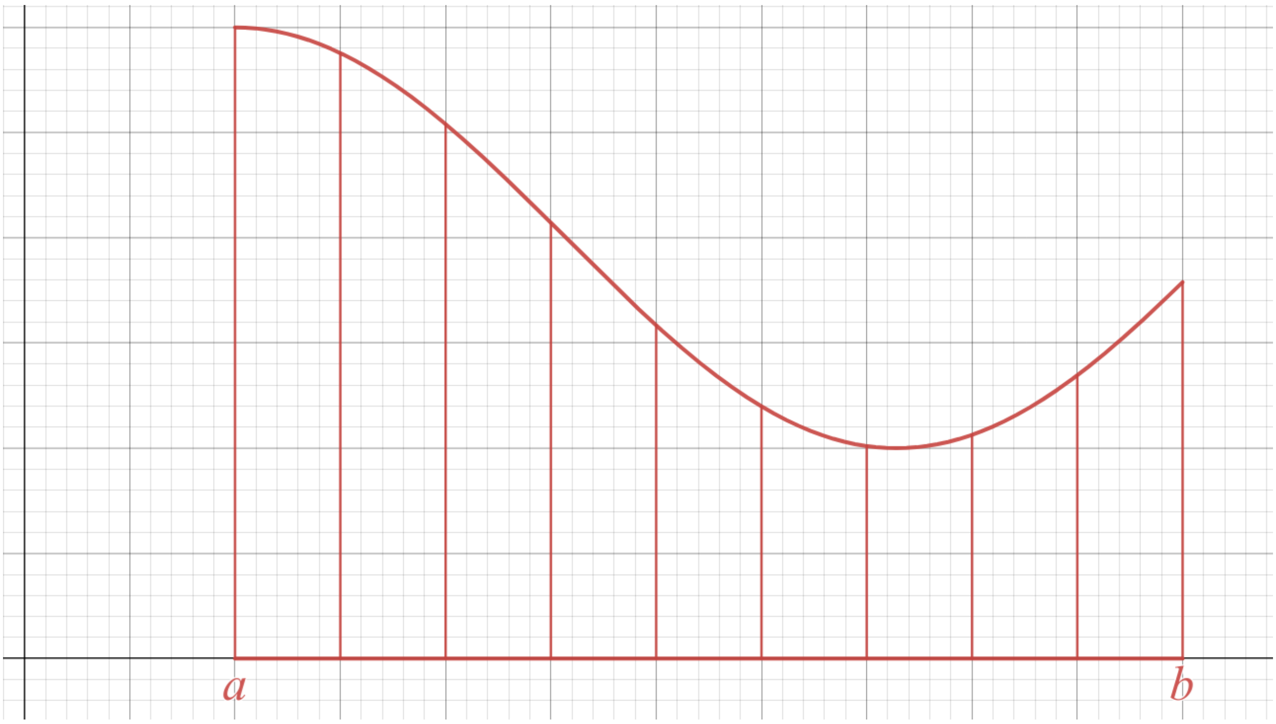
General Definition

Suppose we have a region S under the graph of a function $y = f(x)$ from $x = a$ to $x = b$.



- Divide the interval $[a, b]$ in n subintervals of equal length $\Delta x = (b - a)/n$.



- Select some number x_i^* in each $[x_{i-1}, x_i]$ (can be any number within the subinterval).
- Form the sum: $S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$.

Definite Integral: For a continuous function f, the definite integral of f is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*)\Delta x \right).$$

Important Remarks:

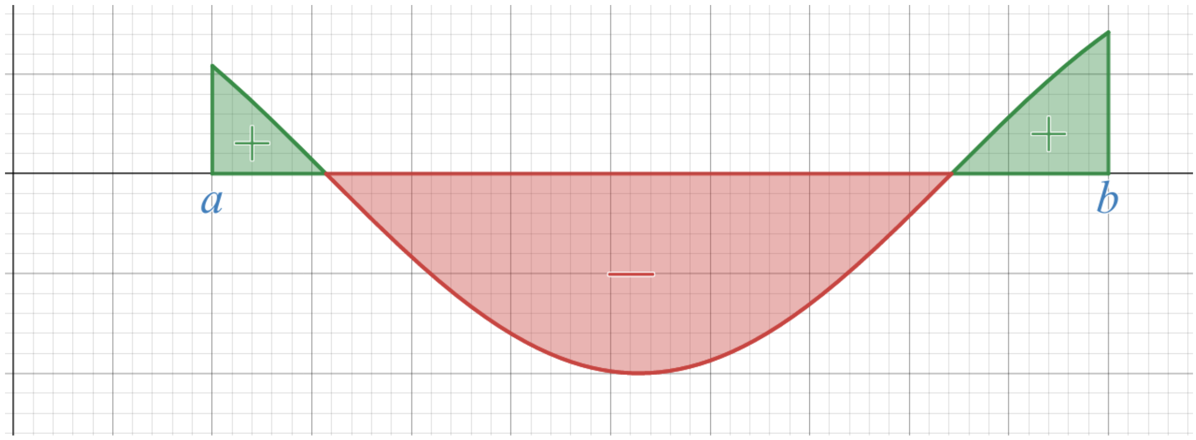
- Description of the terminology:
  - Symbol $\int$:
  - a:
  - b:
  - $f(x)$:
  - dx:
- The definite integral is a **number**! It does not depend on x! This means that

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt = \dots$$

- The expression S_n are called **Riemann Sums**.
- When $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ is the area of the region S:

$$\text{Area}(S) = \int_a^b f(x)\,dx.$$

- If $f(x)$ is negative somewhere, then $\int_a^b f(x)\,dx$ is the **net area** between the graph of $y = f(x)$ and the horizontal line $y = 0$ (the x-axis).



Example 1. Find the value of the following integrals.

(a) $\int_0^1 x\,dx.$ **(b)** $\int_{-1}^1 x\,dx.$ **(c)** $\int_0^2 |x-1|\,dx.$

Useful Trick: Try to intepret the integral geometrically!

Playing with Lower and Upper Bounds

- If we change the order of the lower and upper bounds, then
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$
- If the lower and upper bounds are equal, the definite integral is zero, that is
$$\int_a^a f(x)\,dx = 0.$$

<u>Illustration:</u>

Algebraic operations

For two continuous functions $f(x)$ and $g(x)$ on the interval $[a, b]$,

- <u>Addition</u>: $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
- <u>Substraction:</u> $\displaystyle\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$
- <u>Multiplication by constant</u>: $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

Useful Formulas

Go to Desmos: https://www.desmos.com/calculator/mr9ba23hpz.

- $\int_a^b 1\,dx =$
- $\int_a^b x\,dx =$
- In general,

$$\int_a^b x^n\,dx = \qquad\qquad\qquad .$$

EXAMPLE 2. Using the properties of the integral and the formulas, find the value of the following integrals.

(a) $\int_0^1 2x^2 - x^4\,dx$.

(b) $\int_{-2}^2 4x^4 - 3x^2\,dx$.

Cutting the domain

Let $a < c < b$ and $f(x)$ be a continuous function on $[a, b]$. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Illustration:

Example 3. If it is known that $\displaystyle\int_0^{10} f(x)\,dx = 17$ and $\displaystyle\int_0^{8} f(x)\,dx = 12$, then find $\int_8^{10} f(x)\,dx$.

Comparison Properties

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 4. Use the last comparison property to estimate $\int_1^4 \sqrt{x}\,dx$.