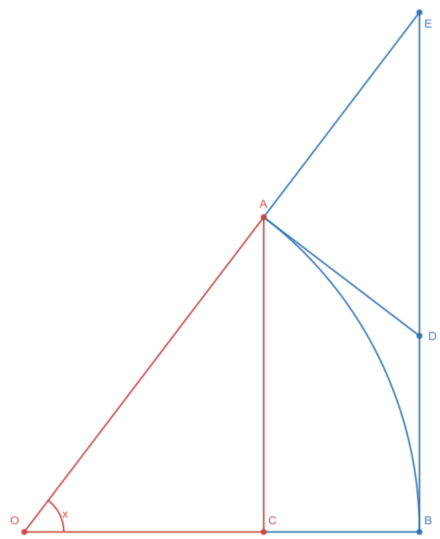
We will prove that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

To prove that, we will use the Squeeze Theorem. Therefore, we need to "squeeze" the function $f(x) = \frac{\sin x}{x}$ between two functions h and g such that $h(x) \leq f(x) \leq g(x)$ and

$$\lim_{x\to 0} h(x) = 1 \quad \text{and} \quad \lim_{x\to 0} g(x) = 1.$$

To find $g(x)$ and $h(x)$, let's consider the following geometric construction:



In red, we have a right triangle $\triangle\, OAC$ with an angle at O of x and a right angle at C. In blue, we have another right triangle $\triangle\, OBE$ with an angle at O of x and a right angle at B. The length of the segment OA is 1 and the length of the segment OB is also 1.

First of all, we see that the length of the segment CA is $\sin x$ and the length of the arc BA is x. We also see that the length of the arc BA is greater that the length of the segment CA. Therefore, we get

$$\sin x \leq x \quad \Rightarrow \quad \frac{\sin x}{x} \leq 1.$$

We let $g(x) = 1$.

Second of all, we see that the segment AD is perpendicular to the segment OE. Also, the length of the arc BA is smaller than the sum of the length of the segments BD and the length of the segment DA. Moreover, the triangle $\triangle\, ADE$ has a right angle at A and its hypothenus is the segment DE. This means that the length of the segment DA is smaller than the length of the segment DE. We then get

$$x \leq \overline{BD} + \overline{DA} \leq \overline{BD} + \overline{DE} = \overline{BE}.$$

However, by the definition of the tangent of x, we have

$$\tan x = \frac{\overline{BE}}{\overline{OB}} = \overline{BE}$$

where the last equality comes from the fact that $\overline{OB} = 1$. Therefore, we get

$$x \leq \overline{BE} = \tan x$$

and since $\tan x = \sin x / \cos x$, this last inequality implies that

$$x \leq \frac{\sin x}{\cos x} \quad \Rightarrow \quad \cos x \leq \frac{\sin x}{x}.$$

So let $h(x) = \cos(x)$.

Finally, we see that

$$\lim_{x \to 0} \cos x = 1 = \lim_{x \to 0} 1.$$

Since $\cos x \leq \frac{\sin x}{x} \leq 1$, by the Squeeze Theorem, we conclude that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$