Math 241 Final Exam, Fall 2013

Name: Solutions.　　ID #: ____________

Section number: ____________

Instructor: ____________

Question	Points	Score
1	20	
2	25	
3	7	
4	8	
5	15	
6	8	
7	7	
8	25	
9	8	
10	7	
11	5	
12	5	
13	10	
Total:	150	

Read all of the following information before starting the exam.

- Electronic devices (calculators, cell phones, computers), books, and notes are not allowed.
- Show all work clearly. You may lose points if we cannot see how you arrived at your solution.
- You do not have to simplify your arithmetic. But be aware that if your answer looks like you need a calculator, you are probably doing it wrong.
- Box or otherwise clearly indicate your final answer.
- This test has 9 pages total including this cover sheet and is worth 150 points. It is your responsibility to make sure that you have all of the pages!

Good luck!

(1) (20 points) Find the limits that exist. If the limit does not exist, write "does not exist".

(a) $\lim_{x\to 2} \frac{x^2-4}{x^2+3x-10}$ $= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{(x+5)\cancel{(x-2)}}$

$= \lim_{x\to 2} \frac{x+2}{x+5}$

$= \frac{2+2}{2+5} = \boxed{\frac{4}{7}}$

(b) $\lim_{x\to 5} \frac{\sqrt{x}-\sqrt{5}}{x-5}$ $= \lim_{x\to 5} \frac{\sqrt{x}-\sqrt{5}}{x-5} \frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}+\sqrt{5}}$

$= \lim_{x\to 5} \frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x}+\sqrt{5})}$

$= \lim_{x\to 5} \frac{1}{\sqrt{x}+\sqrt{5}} = \boxed{\frac{1}{2\sqrt{5}}}$

(c) $\lim_{x\to 0} \frac{\sin x}{x^2+2x}$ $= \lim_{x\to 0} \frac{\sin x}{x(x+2)}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{x+2}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{x+2} = 1 \cdot \frac{1}{2} = \boxed{\frac{1}{2}}$

(d) $\lim_{x\to\infty} \frac{2x^2+\sin x}{x^2-7}$ $= \lim_{x\to\infty} \frac{\cancel{x^2}\left(2+\frac{\sin x}{x^2}\right)}{\cancel{x^2}\left(1-7/x^2\right)}$

$= \lim_{x\to\infty} \frac{2+\frac{\sin x}{x^2}}{1-7/x^2}$

$= \frac{\lim_{x\to\infty}(2+\sin x/x^2)}{\lim_{x\to\infty}(1-7/x^2)} = \frac{2+0}{1-0} = \boxed{2}$

(2) (25 points) Differentiate.

(a) $f(x) = 2x^3 - \sqrt{2} + \dfrac{5}{x^2}$

$$f'(x) = 6x^2 - 0 - \frac{10}{x^3} = 6x^2 - \frac{10}{x^3}$$

(b) $y = 2x^5 \cos x$ $\rightarrow$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^5)\cos x + 2x^5 \frac{d}{dx}\cos x$$

$$= 10x^4 \cos x - 2x^5 \sin x$$

(c) $h(x) = \dfrac{\sin x}{x}$ $\rightarrow$

$$h'(x) = \frac{\frac{d}{dx}(\sin x) \cdot x - \sin x \frac{d}{dx}(x)}{x^2}$$

$$= \frac{\cos x \cdot x - \sin x}{x^2}$$

(d) $m(t) = \sqrt{t^2 + t + 4}$ $\rightarrow$

$$m'(t) = \frac{d}{dt}\left((t^2+t+4)^{1/2} \right)$$

$$= \frac{1}{2}(t^2+t+4)^{-1/2} \cdot \frac{d}{dt}(t^2+t+4)$$

$$= \frac{1}{2}(t^2+t+1) \cdot (2t+1).$$

(e) $k(s) = \tan^2(2s+1)$

$$k'(s) = \frac{d}{ds}\left([\tan(2s+1)]^2 \right)$$

$$= 2\tan(2s+1) \cdot \frac{d}{ds}(\tan(2s+1))$$

$$= 2\tan(2s+1) \cdot \sec^2(2s+1) \cdot \underbrace{\frac{d}{ds}(2s+1)}_{=2}$$

$$= 4\tan(2s+1) \cdot \sec^2(2s+1)$$

(3) (7 points) Find the tangent line to the curve $y = 3x + \frac{1}{x}$ at $x_0 = 1$.

Point-slope: $y - y_0 = m(x - x_0)$

$\uparrow$ derivative $\left.\frac{dy}{dx}\right|_{x=1}$

$x_0 = 1$

$y_0 = 3 \cdot 1 + \frac{1}{1} = 4$

So, $\frac{dy}{dx} = 3 - \frac{1}{x^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 3 - 1 = 2$

So, $y - 4 = 2(x-1)$

$\Rightarrow \boxed{y = 2x + 2}$

(4) (8 points) Find the tangent line to the curve $x^2y - x^4 + y^2 = 1$ at the point $(1, 1)$.

Point-slope: $y - y_0 = m(x - x_0)$

$x_0 = 1$

$y_0 = 1$

$m = \left.\frac{dy}{dx}\right|_{x=1}$

We have

$$\frac{d}{dx}\left(x^2y - x^4 + y^2\right) = \frac{d}{dx}(1)$$

$$\Rightarrow 2xy + x^2\frac{dy}{dx} - 4x^3 + 2y\frac{dy}{dx} = 0$$

$x = x_0 = 1$, $y = y_0 = 1$

$$\Rightarrow 2 \cdot 1 \cdot 1 + 1^2\frac{dy}{dx} - 4(1)^3 + 2(1)\frac{dy}{dx} = 0$$

$$\Rightarrow -2 + 3\frac{dy}{dx} = 0 \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{3}$$

So, $y - 1 = \frac{2}{3}(x-1) \Rightarrow \boxed{y = \frac{2}{3}x + \frac{1}{3}}$

(5) (15 points) Let $f(x) = x^3 - x$.

(a) List the intervals where the graph of $f(x)$ is increasing and decreasing.

$f'(x) = 3x^2 - 1$, C.N. when $f'(x) = 0$ or $f'(x)$ DNE

① $3x^2 - 1 = 0 \iff x = \pm \frac{1}{\sqrt{3}}$ ② $f'(x)$ exists everywhere.

$f'(x) = (\sqrt{3}x+1)(\sqrt{3}x-1)$

factors	$x <$	$-1/\sqrt{3}$	$< x <$	$1/\sqrt{3}$	$< x$
$\sqrt{3}x+1$	$-$		$+$		$+$
$\sqrt{3}x-1$	$-$		$-$		$+$
f'	$+$		$-$		$+$

$f \nearrow$ on $(-\infty, \frac{-1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$

$f \searrow$ on $(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

(b) Find the local maximum and minimum values of $f(x)$.

$x = \frac{-1}{\sqrt{3}}$ is a local max because $f \nearrow$ to $f \searrow$.

& $f(\frac{-1}{\sqrt{3}}) = -\frac{1}{3^{3/2}} + \frac{1}{\sqrt{3}}$.

$x = \frac{1}{\sqrt{3}}$ is a local min because $f \searrow$ to $f \nearrow$

& $f(\frac{1}{\sqrt{3}}) = \frac{1}{3^{3/2}} - \frac{1}{\sqrt{3}}$.

(c) List the intervals where the graph is concave up and concave down.

$f''(x) = 6x$

$f''(x) = 0$

$\Updownarrow$

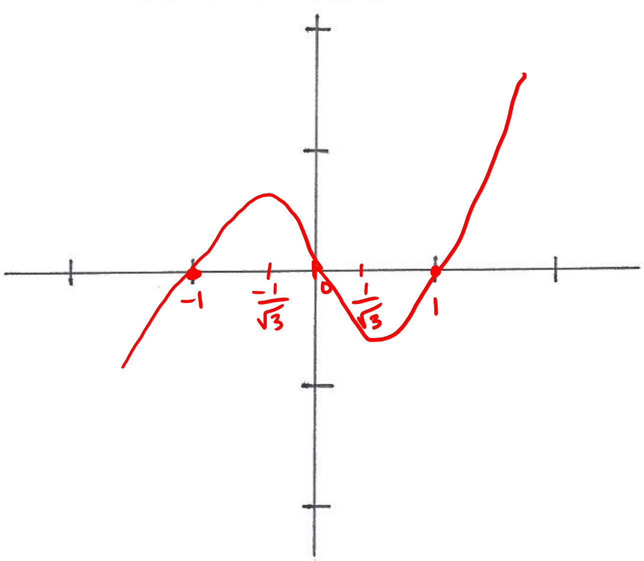
$x = 0$

fact.	$x <$	0	$< x$
$6x$	$-$		$+$
f''	$-$		$+$
f	$\frown$		$\smile$

(d) Sketch the graph of the function.

$x^3 - x = 0 \iff x(x^2 - 1) = 0$

$\iff x = 0,\ x = 1,\ x = -1$



(6) (8 points) Let $Y(N)$ be the yield of an agricultural crop as a function of the nitrogen level N in the soil. A model that is used for this is

$$Y(N) = \frac{N}{1+N^2} \text{ for } N \geq 0,$$

where N is measured in appropriate units. Find the nitrogen level that maximizes the yield.

① Derivative: $y'(N) = \frac{\frac{d}{dN}(N)\cdot(1+N^2) - N\cdot\frac{d}{dN}(1+N^2)}{(1+N^2)^2}$

$= \frac{1+N^2 - N\cdot 2N}{(1+N^2)^2} = \frac{1+N^2-2N^2}{(1+N^2)^2}$

$\Rightarrow \quad y'(N) = \frac{1-N^2}{(1+N^2)^2} = \frac{(1-N)(1+N)}{(1+N^2)^2}$

② Optimize: $y'(N) = 0 \iff 1-N^2 = 0$

$\iff N = \pm 1$

$N \geq 0$ by assumptions $\Rightarrow N = 1$

If $N < 1$, then $1-N > 0 \rightarrow y'(N) > 0 \rightarrow$ increasing

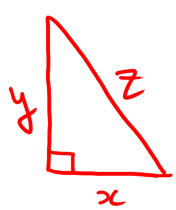
If $N > 1$, then $1-N < 0 \rightarrow y'(N) < 0 \rightarrow$ decreasing

By 1st test, $N=1$ is a max (absolute $N \geq 0$).

③ Answer:

$N = 1$ and $Y(N) = \frac{1}{2}$

(7) (7 points) A right triangle is changing shape. If the base is 3 meters and expanding at 0.2 meters per minute, and the height is 4 meters and shrinking at 0.1 meters per minute, at what rate is the length of the hypotenuse changing?



x: base

y: heigth

z: hypothenuse

Goal:

Find $\frac{dz}{dt} = ??$

Pythagorus

$z^2 = x^2 + y^2 \Rightarrow \frac{d}{dt}(z^2) = \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2)$

$\Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$x = 3$, $y = 4 \rightarrow z = 5$, $\frac{dx}{dt} = 0.2$, $\frac{dy}{dt} = -0.1$

$\Rightarrow 10\frac{dz}{dt} = \frac{6}{5} - \frac{8}{10}$

$\Rightarrow \boxed{\frac{dz}{dt} = \frac{1}{50} \text{ meters/min}}$

(8) (25 points) Integrate.

(a) $\int (x^{\frac{7}{8}} - \sec x \tan x)\,dx = \frac{8}{15} x^{\frac{15}{8}} - \sec x + C$

(b) $\int x \sin(x^2)\,dx$

$u = x^2$
$\to du = 2x\,dx$
$\to \frac{du}{2} = x\,dx$

$\Rightarrow \; = \int \sin(u) \frac{du}{2} = -\frac{\cos(u)}{2} + C$

$= \boxed{-\frac{\cos(x^2)}{2} + C}$

(c) $\int_1^5 \frac{1}{x^2}\,dx = -x^{-1}\Big|_1^5 = -\left(\frac{1}{5} - 1\right)$

$= \boxed{\frac{4}{5}}$

(d) $\int_1^2 (x+1)\sqrt{x^2+2x+2}\,dx =$

$u = x^2+2x+2$
$\to du = (2x+2)\,dx$
$\to \frac{du}{2} = (x+1)\,dx$

$\Rightarrow \int_1^2 \sqrt{u}\,\frac{du}{2} = \frac{2}{3} u^{3/2} \cdot \frac{1}{2} + C$

$= \boxed{\frac{1}{3}\left(x^2+2x+2\right) + C}$

(e) $\int \frac{x}{\sqrt{x+1}}\,dx$

$u = x+1$
$du = dx$

$= \int \frac{u-1}{\sqrt{u}}\,du$

$= \int u^{1/2} - u^{-1/2}\,du$

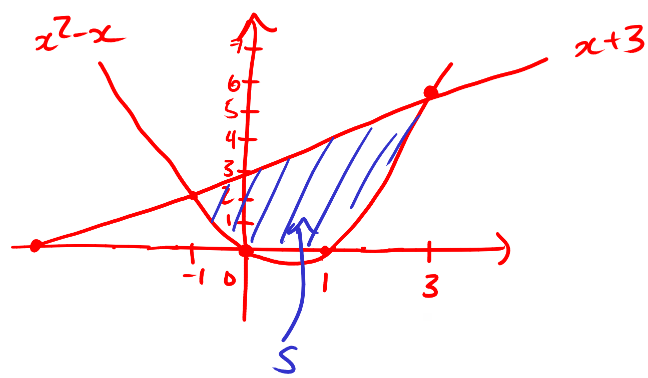
$= \frac{2u^{3/2}}{3} - 2u^{1/2} + C$

$= \boxed{\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C}$

(9) (8 points) Find the area enclosed by the curves $y = x^2 - x$ and $y = x + 3$.

① Picture.

$$x^2 - x = x + 3$$
$$\Leftrightarrow x^2 - 2x - 3 = 0$$
$$\Leftrightarrow (x-3)(x+1) = 0$$
$$\Leftrightarrow x = 3,\ x = -1$$

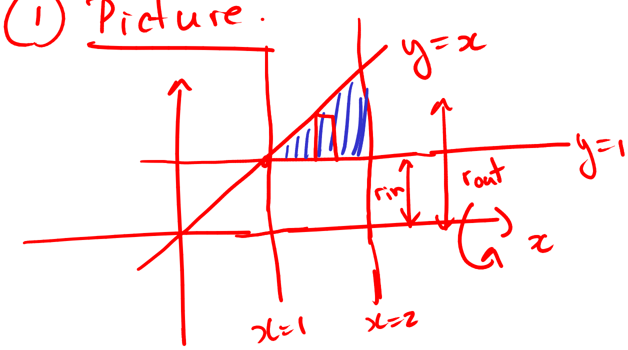


② Integrate

$$\text{Area}(S) = \int_{-1}^{3} x+3-(x^2-x)\,dx$$
$$= \int_{-1}^{3} 2x+3-x^2\,dx$$
$$= x^2 + 3x - \frac{x^3}{3}\Big|_{-1}^{3} = \boxed{29 + \frac{1}{3}}$$

(10) (7 points) Find the volume obtained by rotating about the x–axis the region between $y = 1$, $y = x$, $x = 1$ and $x = 2$.

① Picture.



$r_{in} = 1$

$r_{out} = y = x$

thickness $= dx$

$a = 1$

$b = 2$

② Volume

$$\text{Vol(Solid)} = \int_1^2 \pi\left(r_{out}^2 - r_{in}^2\right)\,dx$$
$$= \int_1^2 \pi\left(x^2 - 1\right)\,dx$$
$$= \pi\left(\frac{x^3}{3} - x\right)\Big|_1^2 = \boxed{\frac{4\pi}{3}}$$

(11) (5 points) (Complete the definition) A function is *differentiable* on the interval $(0,1)$ if ....

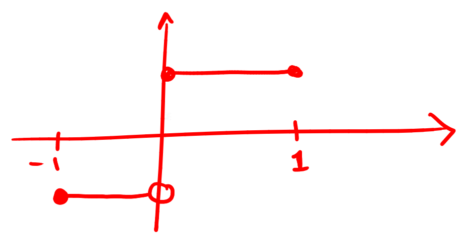
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists for every x inside $(0,1)$.

(12) (5 points) Give an example of a function that is defined on the closed interval $[-1,1]$ but is not continuous.

$$f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

Graph of f:



(13) (10 points) True or false? (You may assume that the functions are defined on the entire real line.)

True (a) Every differentiable function is continuous.

False (b) Every continuous function is differentiable.

True (c) Every integrable function is continuous.

True (d) Every continuous function is integrable.

True (e) For any function $f(x)$, if $f(0) < 0$ and $f(2) > 0$, then $f(c) = 0$ for some $c \in (0,2)$.

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