Math 241 Final Exam, Spring 2013

Name:

Section number:

Instructor:

Question	Points	Score
1	5	
2	5	
3	12	
4	10	
5	17	
6	15	
7	6	
8	12	
9	12	
10	14	
11	17	
12	10	
13	5	
Total:	140	

Read all of the following information before starting the exam.

- Electronic devices (calculators, cell phones, computers), books and notes are not allowed.
- Show all work clearly. You may lose points if we cannot see how you arrived at your solution.
- You do not have to simplify your arithmetic. But be aware that if your answer looks like you need a calculator, you are probably doing it wrong.
- Box or otherwise clearly indicate your final answers.
- This test has 12 pages total including this cover sheet and is worth 140 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (5 points) Compute the following limit

$$\lim_{t\to 4}\left(\frac{t-4}{\sqrt{t}-2}\right)$$

$$= \lim_{t\to 4} \frac{\cancel{(\sqrt{t}-2)}(\sqrt{t}+2)}{\cancel{\sqrt{t}-2}}$$

$$= \lim_{t\to 4} \sqrt{t}+2$$

$$= \sqrt{4} + 2 = \boxed{4}$$

2. (5 points) Compute the following limit, making sure to **justify all steps.**

$$\lim_{x\to\infty}\frac{3x\sin\left(x^2\right)+1}{x^2+4}$$

$$= \lim_{x\to\infty} \frac{3\sin(x^2) + 1/x}{x\,(1+4/x^2)}$$

$$= \lim_{x\to\infty} \frac{3\sin(x^2)}{x(1+4/x^2)} + \lim_{x\to\infty} \frac{1/x}{x(1+4/x^2)}$$

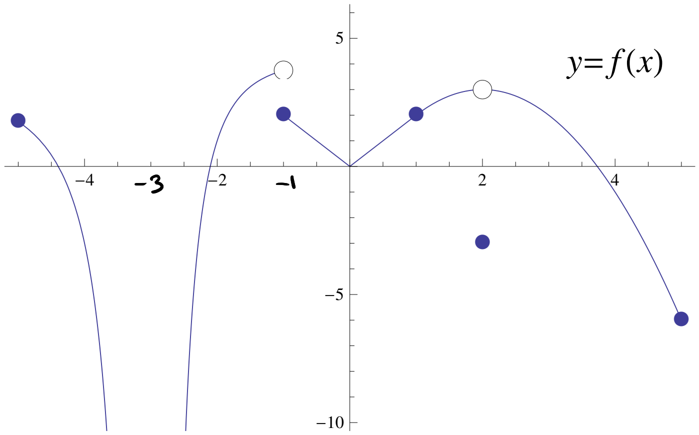
$$= \lim_{x\to\infty} \frac{\sin(x^2)}{x} \cdot \lim_{x\to\infty} \frac{3}{1+4/x^2} + 0$$

Now, $-1 < \sin(x^2) < 1 \Rightarrow -\frac{1}{x} < \frac{\sin(x^2)}{x} < \frac{1}{x}$

Squeeze Theorem: $\lim_{x\to\infty} \frac{\sin(x^2)}{x} = 0.$

So, Answer $= 0 \cdot 3 + 0 = \boxed{0}$

3. The graph of the function $y = f(x)$ is given below:



You will not receive full credit without an explanation at each point.

(a) (4 points) At which points in $[-5, 5]$ does the limit of the function **not exist**? Explain why.

$x = -1$, lim from left $\neq$ limit from right

$x = -3$, lim from left & from right is $-\infty$.

(b) (4 points) At which points in $[-5, 5]$ is the function **discontinuous**? Explain why.

$x = -1$, same reason as in (a)

$x = -3$, f undefined there.

$x = 2$, $\lim_{x \to 2} f(x) \neq f(2)$.

(c) (4 points) At which points in $[-5, 5]$ is the function **not differentiable**? Explain why.

$x = -3$, not continuous $\to$ not differentiable.

$x = -1$, not continuous $\to$ not differentiable.

$x = 2$, not continuous $\to$ not differentiable.

$x = 0$, $f'(0)$ DNE $\left(\lim_{h \to 0} \frac{f(h) - f(0)}{h} \text{ DNE.} \right)$.

4. (10 points) Using the **definition of derivative, NOT differentiation rules**, find $f'(2)$ if

$$f(x) = \frac{1}{x+1}.$$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

$$= \lim_{h \to 0} \frac{3 - (3+h)}{3h(3+h)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{3\cancel{h}(3+h)}$$

$$= \lim_{h \to 0} \frac{-1}{3(3+h)}$$

$$= \boxed{-\frac{1}{9}}$$

5. Compute the following integrals

(a) (5 points) $\int \sin(3x+1)dx$

$u = 3x+1 \longrightarrow du = 3dx$

$$\int \sin(3x+1)\,dx = \int \sin(u)\,\frac{du}{3}$$

$$= \frac{1}{3}\left(-\cos(u)\right) + C$$

$$= -\frac{\cos(3x+1)}{3} + C$$

(b) (5 points) $\int_{-1}^{2} t^3 - t\, dt$

$$= \left(\frac{t^4}{4} - t\right)\Big|_{-1}^{2}$$

$$= \frac{16}{4} - 2 - \left(\frac{1}{4} + 1\right)$$

$$= \frac{16 - 8 - 1 - 4}{4} = \boxed{\frac{3}{4}}$$

(c) (7 points) $\int_0^1 x(x-1)^{2013}dx$

$u = x-1 \longrightarrow du = dx$

$$\int_0^1 x(x-1)^{2013}\,dx = \int_{-1}^{0} (u+1)\,u^{2013}\,du$$

$$= \int_{-1}^{0} u^{2014} + u^{2013}\,du$$

$$= \left(\frac{u^{2015}}{2015} + \frac{u^{2014}}{2014}\right)\Big|_{-1}^{0}$$

$$= 0 - \left(\frac{-1}{2015} + \frac{1}{2014}\right) = \boxed{\frac{1}{2015 \cdot 2014}}$$

6. (15 points) This question refers to the curve defined by

$$2x^3 + xy = y^2.$$

(a) Find dy/dx at the point $(1, 2)$.

$$\frac{d}{dx}\left(2x^3 + xy\right) = \frac{d}{dx}\left(y^2\right)$$

$$\rightarrow \quad 6x^2 + y + x\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\begin{matrix} x=1 \\ y=2 \end{matrix} \rightarrow \quad 6 + 2 + \frac{dy}{dx} = 4\frac{dy}{dx}$$

$$\rightarrow \quad \frac{dy}{dx} = \frac{8}{3}$$

(b) Find the tangent line to the curve at $(1, 2)$.

Point-slope: $\quad y - 1 = \left(\frac{dy}{dx}\right)(x-1)$

$$\frac{dy}{dx} = \frac{8}{3} \quad \rightarrow \quad y - 1 = \frac{8}{3}(x-1)$$

$$\rightarrow \quad y = \frac{8x}{3} - \frac{5}{3}$$

(c) Using the tangent line, estimate the value of y at $x = 1.2$. [You do not have to simplify the arithmetic.]

$$L(x) = \frac{8x}{3} - \frac{5}{3}$$

So, $\quad y(1.2) \approx L(\overset{\frac{6}{5}}{1.2}) = \frac{8 \cdot 6}{3 \cdot 5} - \frac{5}{3}$

$$= \frac{40 - 25}{15}$$

$$= \frac{15}{15} = \boxed{1}$$

7. (6 points) Does the equation $x^3 - 4x^2 + 2 = 1$ have a solution on the interval $[-1, 1]$? Make sure you **justify your answer, and clearly state any theorems that you are using.**

Set $f(x) = x^3 - 4x^2 + 2$.

f continuous, it's a polynomial.

✓ $f(-1) = -1 - 4 + 2 = -3$

× $f(1) = 1 - 4 + 2 = -1$

✓ $f(0) = 0 - 0 + 2 = 2$

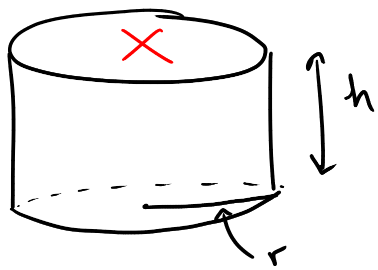
Then 1 is between -3 and 2, so by the IVT, there is a c between -1 & 0 so that

$$c^3 - 4c^2 + 2 = 1$$

8. (12 points) Bob wants to use 12 square inches of metal to make a cylindrical can of height h and radius r. Bob just wants the sides and bottom of the can to be included, but not the top. What is the maximum volume of the can that Bob can make?

Hint: the volume of a cylinder as given is $V = \pi r^2 h$.

Picture:



r: radius (inches) $(r>0)$

h: heigth (inches) $(h>0)$

V: volume (inches^3)

S: surface area (inches^2)

$$S = \text{Area bottom} + \text{Area side}$$
$$= \pi r^2 + 2\pi r h$$

Now, $S = 12 \Rightarrow 12 = \pi r^2 + 2\pi r h$

$$\Rightarrow \frac{12 - \pi r^2}{2\pi r} = h$$

$$\Rightarrow \frac{6}{\pi r} - \frac{r}{2} = h$$

So,

$$V = \pi r^2 h = \pi r^2 \left(\frac{6}{\pi r} - \frac{r}{2} \right) = 6r - \frac{\pi r^3}{2}$$

Optimize:

$$V'(r) = 6 - \frac{3\pi}{2} r^2 = 0 \iff 6 = \frac{3\pi}{2} r^2$$
$$\iff 4\pi = r^2 \iff r = \pm 2\sqrt{\pi}$$

But $r>0$, so $r = 2\sqrt{\pi}$

Now, $V''(r) = -3\pi r < 0$ (because $r>0$).

so $r = 2\sqrt{\pi}$ gives an abs. max.

<u>Answer</u>

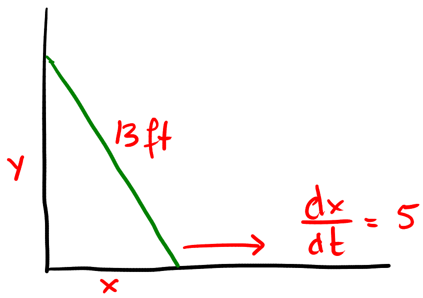
$$\text{max. Volume} = V(2\sqrt{\pi}) = 12\sqrt{\pi} - \frac{8\pi(\sqrt{\pi})^3}{2}$$

$$= 12\sqrt{\pi} - 4\pi^2\sqrt{\pi}$$

$$= \boxed{4\sqrt{\pi}(3 - \pi^2)}$$

9. A 13 foot ladder is leaning against a wall when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

(a) (4 points) How fast is the top of the ladder sliding down the wall at that time?



Goal: $\left.\frac{dy}{dt}\right|_{x=12} = ??$

$$x^2+y^2 = 13 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$x = 12 \rightarrow y = \sqrt{13^2 - 12^2} = \sqrt{25} = 5 \Rightarrow \boxed{\frac{dy}{dt} = -12 \text{ ft/sec}}$$

(b) (4 points) At what rate is the area of the triangle formed by the ladder, wall and ground changing at that time?

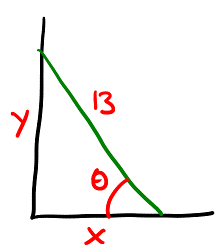
$$\text{Area} = x \cdot y \Rightarrow \frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 5 \cdot 5 + 12 \cdot (-12)$$

$$= 25 - 144$$

$$\Rightarrow \boxed{\frac{dA}{dt} = -121 \text{ ft}^2/\text{sec}}$$

(c) (4 points) At what rate is the angle θ between the ladder and the ground changing at that time?



$$\cos\theta = \frac{x}{13} \rightarrow \frac{d}{dt}(\cos\theta) = \frac{1}{13}\frac{dx}{dt}$$

$$\rightarrow -\sin(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$$

$x = 12,\ y = 5$

$\sin\theta = \frac{y}{13} = \frac{5}{13}$

$$\rightarrow -\frac{5}{13} \cdot \frac{d\theta}{dt} = \frac{1}{13} \cdot 5$$

$$\rightarrow \boxed{\frac{d\theta}{dt} = -1 \text{ rad/sec}}$$

10. Let $f(x) = x^4 - 4x^3 + 10$. Then $f'(x) = 4x^2(x-3)$, $f''(x) = 12x(x-2)$.

(a) (4 points) Find the intervals where f is increasing, and those where it is decreasing.

$f'(x) = 0 \iff x = 0$ or $x = 3$

	$x <$	0	$< x <$	3	$< x$
$4x^2$	$+$		$+$		$+$
$x-3$	$-$		$-$		$+$
f'	$-$	0	$-$	0	$+$
f	$\searrow$		$\searrow$		$\nearrow$

(b) (4 points) Find the local and global maxima and minima. Give both the values, and where they are attained.

C.N.: $x=0$, $x=3$

$x=0$

No local max/min

$x=3$

1st test, local min.

$f(3) = -17$

(c) (4 points) Find the intervals where f is concave up, and those where it is concave down.

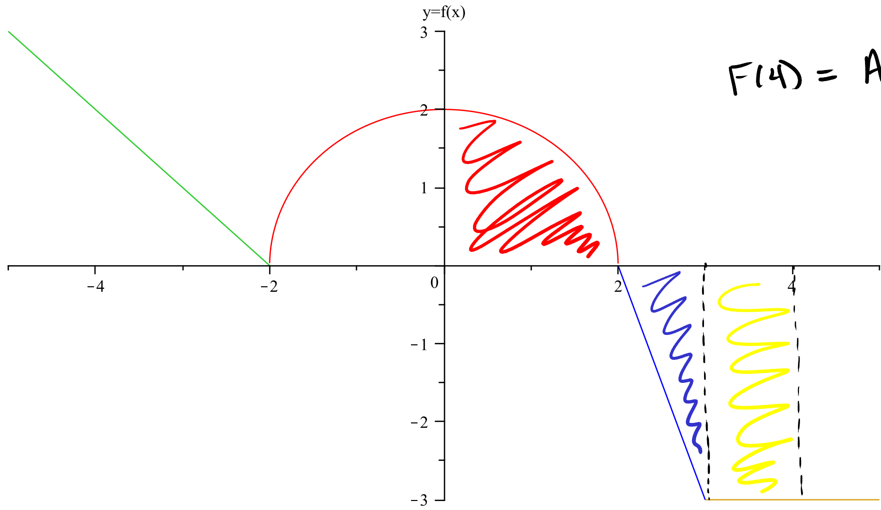
$f''(x) = 0 \iff x = 0$, $x = 2$

factors	$x <$	0	$< x <$	2	$< x$
$12x$	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

(d) (2 points) Find the inflection points of f.

Inflection pts are $x=0$, $x=2$

11. Below is a plot of a function $y = f(x)$. It is linear on the intervals $[-5,-2]$, $[2,3]$, and $[3,5]$, and a radius 2 semicircle on the interval $[-2,2]$:



$F(4) = \text{Area}(\;) - \text{Area}(\;3) - \text{Area}(\;3)$

Define $F(x) = \int_0^x f(t)\,dt$ for x in the interval $[-5,5]$.

(a) (8 points) Evaluate the following

$$F(4) = \frac{\pi 2^2}{2} - \frac{3\cdot 1}{2} - 3\cdot 1 = \pi - 4.5$$

FTC I $F'(4) = f(4) = -3$

$$F''(4) = f'(4) = 0$$

$$F(-2) = \text{Area}(\;2) = \frac{\pi \cdot 2^2}{4} = \pi$$

(b) (4 points) On what interval(s) is the function F increasing?

$F'(x) = f(x)$.

F Increases when $f(x) > 0$: on $(-5, 2)$

F Decreases when $f(x) < 0$: on $(2, 5)$

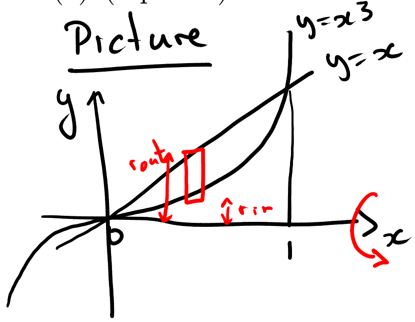
(c) (5 points) Another function $G(x)$ is defined as $G(x) = \int_0^{x^2} f(t)dt$. Find $G'(2)$.

$$G'(x) = f(x^2) \cdot \frac{d}{dx}(x^2) = 2x\, f(x^2).$$

So, $G'(2) = 2\cdot 2\cdot f(4) = 4\cdot(-3) = \boxed{-12}$

12. Let $f(x) = x$ and $g(x) = x^3$. Let R be the region between the graphs of f and g as x ranges over the interval $[0, 1]$.

(a) (5 points) Find the volume of the region arrived at by rotating R about the x-axis.



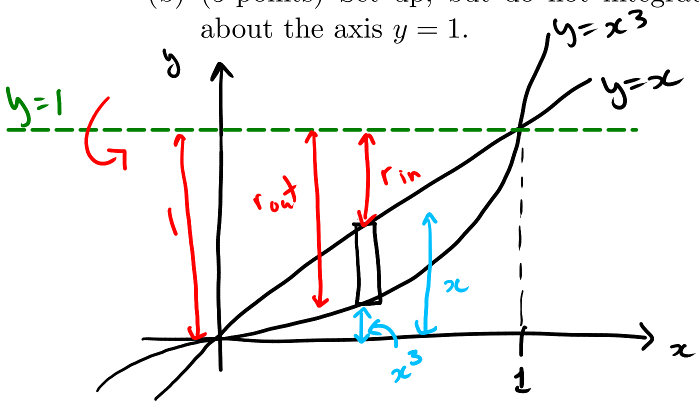
$r_{out} = x$ thickness $= dx$

$r_{in} = x^3$

Volume.

$$\text{Vol(Solid)} = \int_0^1 \pi r_{out}^2 - \pi r_{in}^2 \, dx$$
$$= \pi \int_0^1 x^2 - x^6 \, dx$$
$$= \pi \left(\frac{x^3}{3} - \frac{x^7}{7} \right) \Big|_0^1$$
$$= \pi \left(\frac{1}{3} - \frac{1}{7} \right) = \boxed{\frac{4\pi}{21}}$$

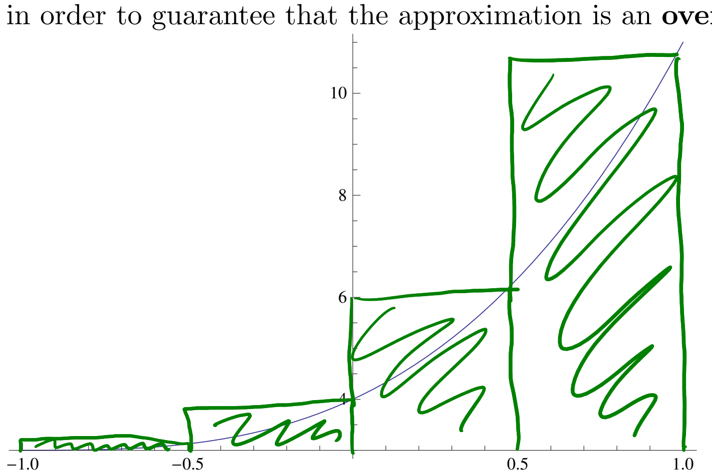
(b) (5 points) Set up, but do not integrate, an integral for the volume when R is rotated about the axis $y = 1$.



$$\text{Vol(Solid)} = \int_0^1 \pi r_{out}^2 - \pi r_{in}^2 \, dx$$
$$= \boxed{\int_0^1 \pi (1-x^2)^2 - \pi (1-x)^2 \, dx.}$$

$r_{in} = 1 - x$, $r_{out} = 1 - x^2$, thick. $= dx$

13. (5 points) Leslie would like to approximate the area under the following curve on the interval $[-1, 1]$ using a Riemann sum and intervals of length 0.50. Sketch the areas she should compute in order to guarantee that the approximation is an **overestimate**.



$\Delta x = 0.5$, $a = -1$, $b = 1$.

$x_1 = -0.5$

$x_2 = 0$

$x_3 = 0.5$

$x_4 = 1$