Math 241/251A Final Exam

Your name: Solution

Select your instructor and section time:

- ☐ Luca Candelori (Thursday 1:30pm)
- ☐ Luca Candelori (Friday 10:30am)
- ☐ Erik Guentner (Wednesday 8:30am)
- ☐ Asaf Hadari (Thursday 10:30am)
- ☐ Piper Harron (Thursday 12:00pm)
- ☐ Piper Harron (Friday 9:30am)
- ☐ Mushfeq Khan (Wednesday 10:30)
- ☐ Mushfeq Khan (Wednesday 1:30pm)
- ☐ Daisuke Takagi (Thursday 1:30pm)
- ☐ Daisuke Takagi (Friday 11:30am)
- ☐ David Webb (Friday 8:30)
- ☐ David Yuen (Thursday 8:30am)
- ☐ David Yuen (Thursday 10:30am)

1 (16)	
2 (4)	
3 (10)	
4 (15)	
5 (3)	
6 (10)	
7 (12)	
8 (10)	
9 (10)	
10 (10)	
11 (18)	
12 (6)	
13 (6)	
14 (10)	
TOTAL (140)	

Justify all your work. Answers without suitable justification will receive no credit.

Problem 1. (16 points) Evaluate the following limits. If the limit is infinite, indicate whether it is ∞ or $-\infty$. (Do not use l'Hôspital's rule.)

a. $\lim\limits_{x\to\infty} \dfrac{x^3+x}{3x^3-1}$

Greatest power: x^3 $\to$ Divide coefficients in front of x^3

$\to$ $\boxed{\text{Answer} = \frac{1}{3}}$

b. $\lim\limits_{x\to 2^+} \dfrac{4-2x}{|2x-4|}$

$x \to 2^+ \to x > 2 \to |2x-4| = |2(x-2)| = 2(x-2)$

So, $\lim\limits_{x\to 2^+} \dfrac{4-2x}{|2x-4|} = \lim\limits_{x\to 2^+} \dfrac{4-2x}{2(x-2)}$

$= \lim\limits_{x\to 2^+} -\dfrac{x-2}{x-2} = \boxed{-1}$

c. $\lim\limits_{x\to 0} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 0} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= \lim\limits_{x\to 0} x+2$

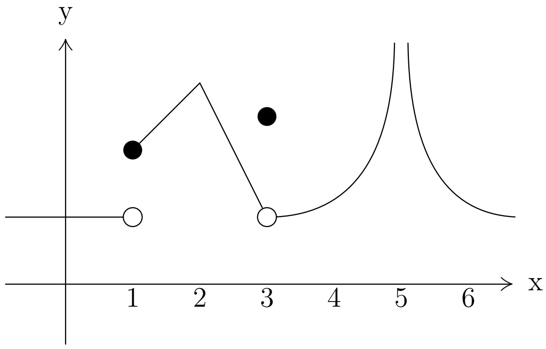
$= \boxed{2}$

d. $\lim\limits_{\theta\to 0} \dfrac{\sin(2\theta)}{\theta} = \lim\limits_{\theta\to 0} \dfrac{\sin(2\theta)-\sin(0)}{\theta - 0}$

$= \dfrac{d}{d\theta}\left(\sin(2\theta)\right)\Big|_{\theta=0}$

$= \cos(2\theta)\cdot 2 \Big|_{\theta=0} = \boxed{2}$

Problem 2. (4 points) Below is the graph of $y = f(x)$.



a. Find the values of a for which $\lim_{x \to a^+} f(x)$ is infinite or does not exist.

$a = 5$.

b. Find the values of a for which $\lim_{x \to a^-} f(x)$ is infinite or does not exist.

$a = 5$.

c. Find the values of a for which $\lim_{x \to a} f(x)$ is infinite or does not exist.

$a = 1$, $a = 5$.

d. Find the values of a for which f is not continuous at $x = a$.

$a = 1$, $a = 3$, $a = 5$.

Problem 3. (10 points)

a. State the definition of $f'(x)$ as a limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

b. Let $f(x) = \sqrt{2x}$. Use the definition of the derivative to calculate $f'(2)$ (do not use differentiation rules).

$$f'(2) = \lim_{h \to 0} \frac{\sqrt{2(2+h)} - \sqrt{2 \cdot 2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4+2h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{4+2h} - 2)}{h} \frac{(\sqrt{4+2h} + 2)}{(\sqrt{4+2h} + 2)}$$

$$= \lim_{h \to 0} \frac{4 + 2h - 4}{h(\sqrt{4+2h} + 2)}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{4+2h} + 2)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{4+2h} + 2} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

Problem 4. (15 points) Find the following derivatives using differentiation rules. You do not have to simplify your answers.

a. $\frac{d}{dx}\Big(\sin(x)\tan(x^2)\Big)$

$$= \frac{d}{dx}(\sin x)\tan(x^2) + \sin x \frac{d}{dx}(\tan(x^2))$$

$$= \cos(x)\tan(x^2) + \sin(x)\sec^2(x^2)\cdot 2x$$

$$= \cos(x)\tan(x^2) + 2x\sin(x)\sec^2(x^2).$$

b. $\frac{d}{dx}\Big(\frac{x}{x^3-1}\Big)$

$$= \frac{\frac{d}{dx}(x)\,(x^3-1) - x\frac{d}{dx}(x^3-1)}{(x^3-1)^2}$$

$$= \frac{(1)(x^3-1) - x(3x^2)}{(x^3-1)^2} = \frac{x^3-1-3x^3}{(x^3-1)^2} = \boxed{-\frac{2x^3+1}{(x^3-1)^2}}$$

c. $\frac{d}{dx}\Big(\sqrt{\cos(2x+1)}\Big)$

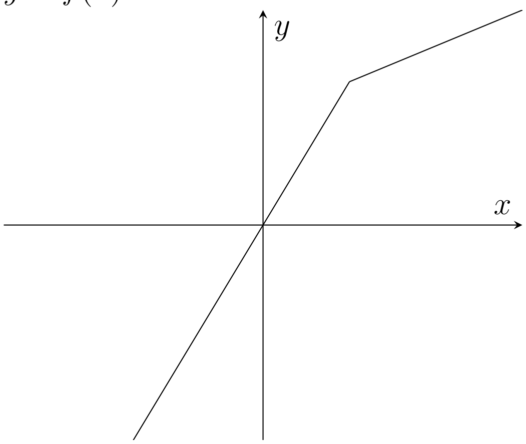
$$= \frac{1}{2\sqrt{\cos(2x+1)}}\cdot\frac{d}{dx}\Big(\cos(2x+1)\Big)$$

$$= \frac{1}{2\sqrt{\cos(2x+1)}}\Big(-\sin(2x+1)\cdot\frac{d}{dx}(2x+1)\Big)$$

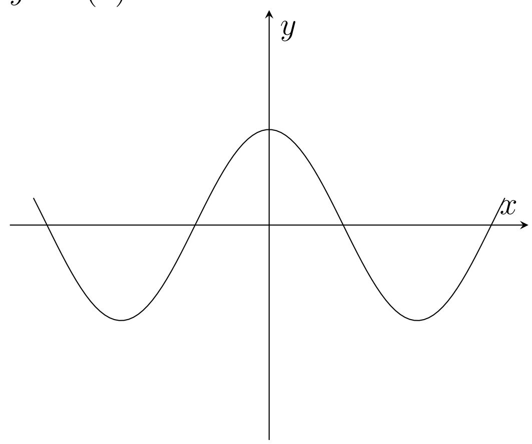
$$= -\frac{2\sin(2x+1)}{2\sqrt{\cos(2x+1)}} = \boxed{-\frac{\sin(2x+1)}{\sqrt{\cos(2x+1)}}}$$

Problem 5. (3 points) Decide which function on the left has which derivative on the right.

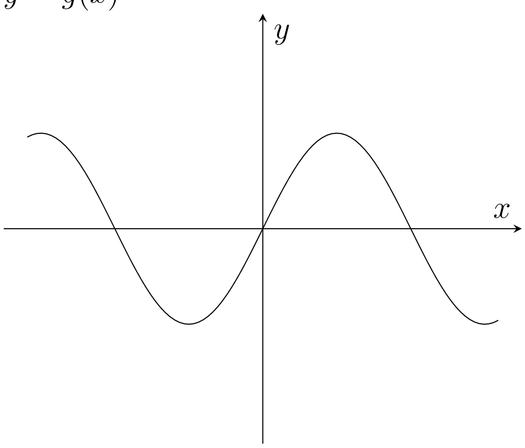
1. $y = f(x)$



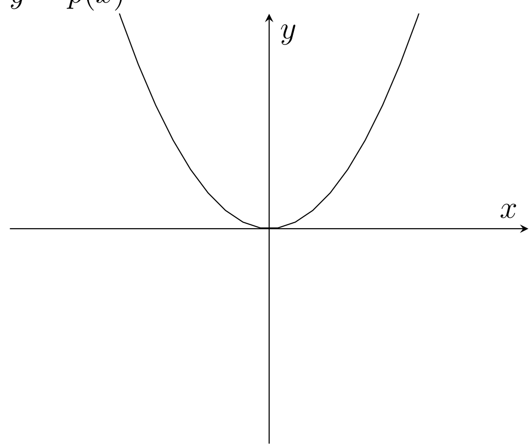
a. $y = k(x)$



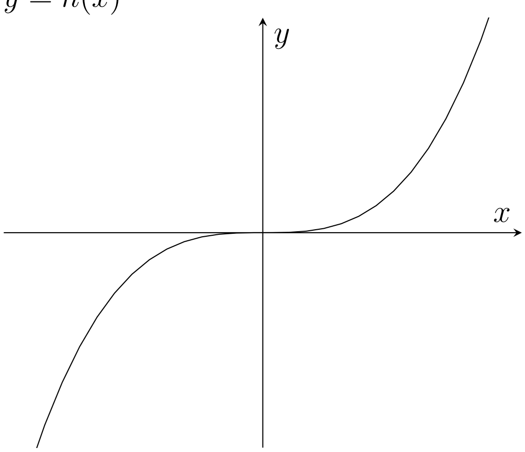
2. $y = g(x)$



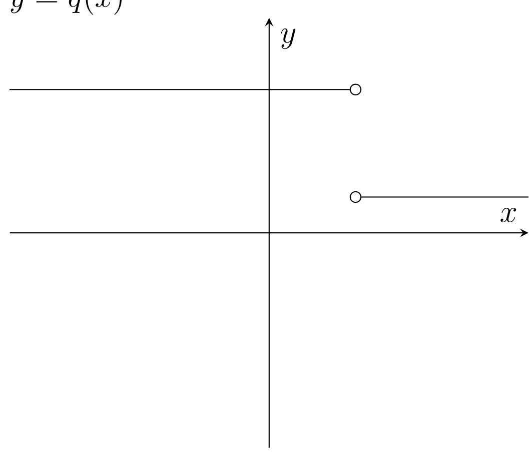
b. $y = p(x)$



3. $y = h(x)$

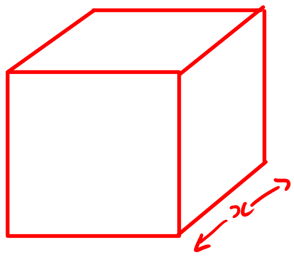


c. $y = q(x)$



1. $f'(x) =$ ☐ $k(x)$ ☐ $p(x)$ ☒ $q(x)$
2. $g'(x) =$ ☒ $k(x)$ ☐ $p(x)$ ☐ $q(x)$
3. $h'(x) =$ ☐ $k(x)$ ☒ $p(x)$ ☐ $q(x)$

Problem 6. (12 points) A cube of ice is melting evenly at a rate of 12 $\mathrm{cm}^3/\mathrm{hour}$. How fast is the side length of the cube changing when the side length is 4 cm?



V: volume of the cube (cm^3)

x: side length of the cube (cm)

$\frac{dV}{dt} = -12\ \mathrm{cm}^3/\mathrm{hour}$.

Goal: $\left.\frac{dx}{dt}\right|_{x=4} = ??$

$$V = x^3 \implies \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\implies -12 = 3 \cdot 4^2 \frac{dx}{dt}$$

$$\implies \frac{-12}{3 \cdot 4^2} = \frac{dx}{dt}$$

$$\implies \frac{dx}{dt} = -\frac{1}{4} = -0.25.$$

So, the length is changing at a rate of $-0.25\ \mathrm{cm}/\mathrm{h}$.

Problem 7. (12 points) Let $f(x) = x^4 - 2x^3$.

a. Find the critical points of f and classify them as local minima, local maxima or neither.

$f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$

So, $f'(x) = 0 \iff x = 0$ or $2x - 3 = 0 \iff x = 0$ or $x = \frac{3}{2}$

So, C.N: $0, \frac{3}{2}$.

Now, $f''(x) = 12x^2 - 12x \Rightarrow$ $\underline{x=0}$ $f''(0) = 0$, $\underline{\text{neither}}$

$\underline{x = \frac{3}{2}}$ $f''(\frac{3}{2}) = 9 > 0$, $\underline{\text{local min.}}$

b. On which intervals is f increasing and on which is f decreasing?

factors	$x <$	0	$< x <$	$\frac{3}{2}$	$< x$
$2x^2$	$+$	0	$+$	$+$	$+$
$2x-3$	$-$	$-$	$-$	0	$+$
f'	$-$	0	$-$	0	$+$
f	$\searrow$		$\searrow$		$\nearrow$

c. Find the inflection points of f and the intervals on which it is concave up and those on which it is concave down.

$f''(x) = 12x^2 - 12x = 12x(x-1)$

So, $f''(x) = 0 \iff x = 0$ or $x = 1$

factors	$x <$	0	$< x <$	1	$< x$
$12x$	$-$		$+$		$+$
$x-1$	$-$		$-$		$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

$\underline{\text{Inflection Points:}}$

$x = 0$

$x = 1$

d. Find the absolute maximum and the absolute minimum of f on the interval $[-1, 1]$.

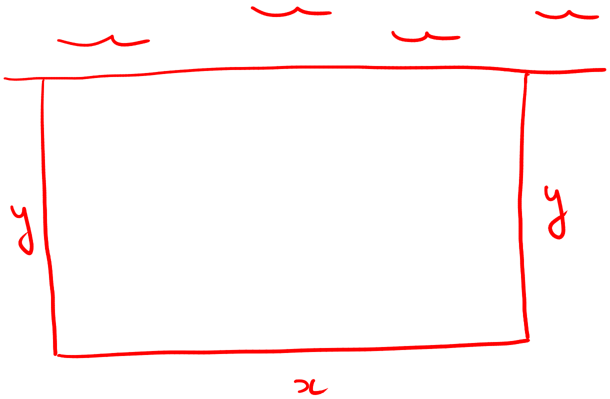
$\underline{\text{C.N. inside } [-1,1]}$: $x = 0$

$\underline{\text{Values of } f \text{ at C.N.}}$: $f(0) = 0$

abs. min $= \min\{f(0), f(-1), f(1)\} = \min\{0, 3, -1\} = -1$

abs. max $= \max\{f(0), f(-1), f(1)\} = \max\{0, 3, -1\} = 3$

Problem 8. (10 points) A rectangular section of a beach reserved for monk seals is being fenced off on three sides (the fourth side borders on the ocean and does not require fencing). If there are $100m$ of fencing, what is the largest area that can be fenced off?



x: width of beach (m)

y: heigth of beach. (m)

P: fencing (m)

A: area of the portion of the beach (m^2)

Goal: optimize A.

We have $A = xy$ & $P = 2y + x = 100$

So, $x = 100 - 2y$

$\Rightarrow A = (100-2y)y$

$\Rightarrow A(y) = 100y - 2y^2$

Now, $A'(y) = 100 - 4y = 0 \iff y = 25$

Also, $A''(y) = -4 \xRightarrow[\text{test}]{2^{nd}}$ means an absolute max. at $y = 25$

So, $\boxed{y = 25m, \; x = 100 - 2y = 50m, \quad A = 1250\,m^2}$

Problem 9. (10 points) Find an equation for the tangent line to the curve $x^2y^2 = 9$ at the point $(3, -1)$.

Point - slope: $y - (-1) = m(x-3)$ ← derivative at $x=3$ $= \frac{dy}{dx}$

We have $\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(9) \Rightarrow \frac{d}{dx}(x^2)y^2 + x^2\frac{d}{dx}(y^2) = 0$

$\Rightarrow 2xy^2 + 2x^2y\frac{dy}{dx} = 0$

So, $\frac{dy}{dx} = -\frac{2xy^2}{2x^2y} = -\frac{y}{x} \underset{y=-1}{\overset{x=3}{\Longrightarrow}} \frac{dy}{dx} = -\frac{(-1)}{3} = \frac{1}{3}$

So, $\boxed{y+1 = \frac{1}{3}(x-3) \quad \text{or} \quad y = \frac{x}{3} - 2}$

Problem 10. (10 points) Show that $f(x) = 2x - \cos(x)$ has exactly one zero in the interval $[-\pi, \pi]$.

a. Show that $f(x)$ has a zero.

$f(-\pi) = 2(-\pi) - \cos(-\pi) = -2\pi + 1 < 0$

$f(\pi) = 2\pi - \cos(\pi) = 2\pi + 1 > 0$

f is continuous $\Longrightarrow$ Intermediate Value Theorem implies there is c, $-\pi < c < \pi$, such that $2c - \cos(c) = 0$.

b. Use Rolle's Theorem to show that it has exactly one zero.

Suppose there is another one, say c_2, such that $f(c_2) = 0$ (we argue by contradiction)

Then, $f(c) = f(c_2) = 0$ and f is differentiable

Rolle's theorem $\Longrightarrow$ there is a d between c and c_2 such that $f'(d) = 0$.

But $f'(x) = 2 + \sin(x) > 0$ because $-1 \le \sin x \le 1$.

So $f'(d)$ can't, in reality, be 0.

So, by the method of proof by contradiction, there is only one zero.

Problem 11. (18 points) Evaluate the following integrals.

a. $\displaystyle\int_0^1 2x\sqrt{x^2+3}\,dx$

$u = x^2+3 \Rightarrow du = 2x\,dx$

So, $\displaystyle\int_0^1 2x\sqrt{x^2+3}\,dx = \int_3^4 u^{1/2}\,du$

$= \frac{2}{3}u^{3/2}\Big|_3^4$

$= \frac{2}{3}\left(4^{3/2} - 3^{3/2}\right).$

b. $\displaystyle\int \sin^2(x)\cos(x)\,dx$

$u = \sin x \rightarrow du = \cos(x)\,dx$

So, $\displaystyle\int \sin^2(x)\cos(x)\,dx = \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \boxed{\frac{\sin^3(x)}{3} + C}$

c. Find $f(x)$ such that $f'(x) = \dfrac{2}{x^2}$ and $f(1) = 0$.

$f'(x) = 2x^{-2} \Rightarrow f(x) = \frac{2x^{-1}}{-1} + C = C - \frac{2}{x}.$

But $f(1) = 0 \Rightarrow 0 = f(1) = C - 2 \Rightarrow C = 2$

Thus, $\boxed{f(x) = 2 - \frac{2}{x}}$

Problem 12. (6 points) Setup an integral for the area between the curve $y = x^2 + 2x + 1$ $= (x+1)^2$ and the line $y = x + 1$. You do not need to evaluate the integral.

<u>Points of intersection</u>

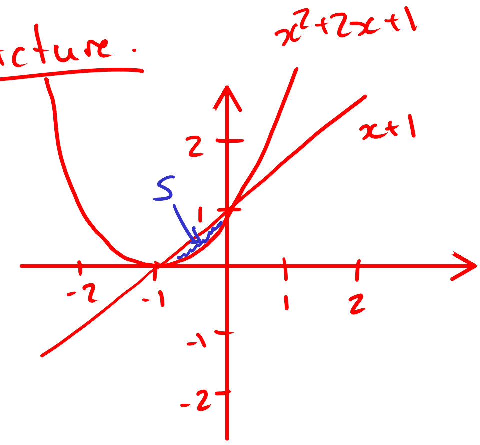
$x^2 + 2x + 1 = x + 1$

$\Leftrightarrow \quad x^2 + x = 0$

$\Leftrightarrow \quad (x+1)x = 0$

$\Leftrightarrow \quad x = -1 \text{ or } x = 0$

<u>Picture.</u>



<u>Integral:</u> $a = -1,\ b = 0,\ g(x) = x^2 + 2x + 1,\ f(x) = x + 1$

$$\text{Area}(S) = \int_{-1}^{1} x + 1 - x^2 - 2x - 1 \, dx$$

Problem 13. (6 points) Estimate $\int_{-1}^{2} (x^2 + 1)\, dx$ with a Riemann sum using left endpoints of 3 equal subintervals.

$a = -1$

$b = 2$

$n = 3$

$$\Delta x = \frac{b-a}{n} = \frac{2+1}{3} = 1$$

$x_0 = a + 0 \cdot \Delta x = -1 + 0 \cdot 1 = -1$

$x_1 = a + 1 \cdot \Delta x = -1 + 1 \cdot 1 = 0$

$x_2 = a + 1 \cdot \Delta x = -1 + 2 \cdot 1 = 1.$

L_3 $\quad h_0 = f(x_0) = 2$

$h_1 = f(x_1) = 1$

$h_2 = f(x_2) = 2$

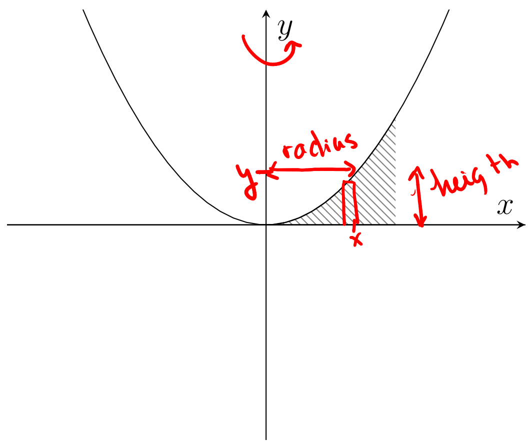
So,

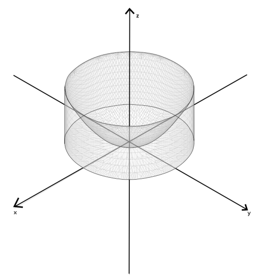
$$\int_{-1}^{2} x^2 + 1 \, dx \approx h_0 \cdot \Delta x + h_1 \cdot \Delta x + h_2 \cdot \Delta x$$

$$= 2 \cdot 1 + 1 \cdot 1 + 2 \cdot 1$$

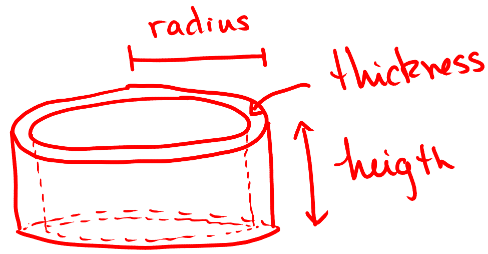
$$= \boxed{5}$$

Problem 14. (10 points) Consider the region between $y = x^2$, the x-axis and the line $x = 1$. Find the volume of the solid that is formed by rotating that region around the y-axis.





Cylindrical shells:



radius $= x$

heigth $= y = x^2$

thickness $= dx$

$$\text{Vol(Solid)} = \int_a^b 2\pi \,(\text{radius})(\text{heigth}) \ \text{thickness}$$

$$= \int_0^1 2\pi \, x \, x^2 \, dx$$

$$= 2\pi \int_0^1 x^3 \, dx$$

$$= 2\pi \left. \frac{x^4}{4} \right|_0^1$$

$$= 2\pi \frac{1}{4}$$

$$= \boxed{\frac{\pi}{2}}$$