Pierre

1. Calculate the following limits. Do **not** use l'Hôpital's rule. If the limit is infinite, specify whether it is $+\infty$ or $-\infty$.

(a) (4 points) $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4}$

$$\frac{x^2-4x}{x^2-3x-4} = \frac{x\cancel{(x-4)}}{\cancel{(x-4)}(x+1)} = \frac{x}{x+1}$$ (2 pts simplify)

So, $$\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4}\frac{x}{x+1} = \frac{\lim_{x\to 4} x}{\lim_{x\to 4} x+1} = \boxed{\frac{4}{5}}$$

(1 pt. quotient rule) (1pt answer)

(b) (4 points) $\lim_{x\to 3^-} \frac{x^2-9}{|x-3|}$

$x<3 \implies |x-3| = 3-x$. (1pt.)

$x^2-9 = (x+3)(x-3)$. → (1pt.)

So, $$\lim_{x\to 3^-}\frac{x^2-9}{|x-3|} = \lim_{x\to 3^-}\frac{(x+3)\cancel{(x-3)}}{-\cancel{(x-3)}} = \lim_{x\to 3^-} -(x+3) = \boxed{-6}$$

(1pt simplify) (1pt. answer)

(c) (4 points) $\lim_{x\to 0} \frac{x}{5x^2+6\sin x}$

$$\lim_{x\to 0}\frac{x}{5x^2+6\sin x} = \lim_{x\to 0}\frac{1}{5x+6\frac{\sin x}{x}}$$ (1pt. simpl. x)

(1pt. quotient rule) → $$= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} 5x + 6\lim_{x\to 0}\frac{\sin x}{x}} = \boxed{\frac{1}{6}}$$ (1pt Answer)

(1pt. =1)

(d) (4 points) $\lim_{x\to\infty} \frac{3x+1}{\sqrt{2x^2+\cos x}}$

If squared only got 9/2 2pts. Multiplied by $\sqrt{2x^2+\cos x}$ 1pt.

$$\lim_{x\to\infty}\frac{3x+1}{\sqrt{2x^2+\cos x}} = \lim_{x\to\infty}\frac{x\,(3+1/x)}{x\sqrt{2+\frac{\cos x}{x^2}}}$$ (Factor x 1pt.)

$$= \lim_{x\to\infty}\frac{3+1/x}{\sqrt{2+\frac{\cos x}{x^2}}}$$

Squeeze Thm: (1pt.)

$$-\frac{1}{x^2} \le \frac{\cos x}{x^2} \le \frac{1}{x^2}$$

$$\Rightarrow 0 \le \lim_{x\to\infty}\frac{\cos x}{x^2} \le 0$$

$$\Rightarrow \lim_{x\to\infty}\frac{\cos x}{x^2} = 0.$$

$$= \frac{3+0}{\sqrt{2+0}} = \boxed{\frac{3}{\sqrt{2}}}$$ (Answer 1pt.)

(Quotient Rule & Root rule 1pt.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ (1 pts)

2. (6 points) Let $f(x) = \dfrac{1}{x^2}$. Using the definition of the derivative as a limit, calculate $f'(3)$. (Warning: You will not get credit if you use the rules of differentiation.)

Julian

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$ (1 pts)

$$= \lim_{h\to 0} \frac{x^2 - (x^2+2xh+h^2)}{h(x+h)^2 x^2}$$ (1 pt)

$$= \lim_{h\to 0} -\frac{2x+h}{x^2(x+h)^2}$$ (1 pt)

$$= -\frac{2x}{x^4} = -\frac{2}{x^3}$$ (1 pt)

$$f'(3) = -\frac{2}{3^3} = -\frac{2}{27}$$ (1 pt)

$\boxed{\text{OR}}$

$$f'(3) = \lim_{h\to 0} \frac{\frac{1}{(3+h)^2} - \frac{1}{3^2}}{h}$$ (1 pts)

$$= \lim_{h\to 0} \frac{9-(9-6h+h^2)}{h(3+h)^2 9}$$ (1 pt)

$$= \lim_{h\to 0} -\frac{6+h}{(3+h)^2 9}$$ (1 pt)

$$= -\frac{6}{3^2 \cdot 9} = -\frac{2}{27}$$ (2 pts)

−1/2 for small errors, but work is consistent after

−5 if definition of derivative is significantly wrong.

i.e $f(x+h) = \frac{1}{x^2}+h$, $\frac{1}{x^2+h}$, etc.

no points off for missing limit signs.

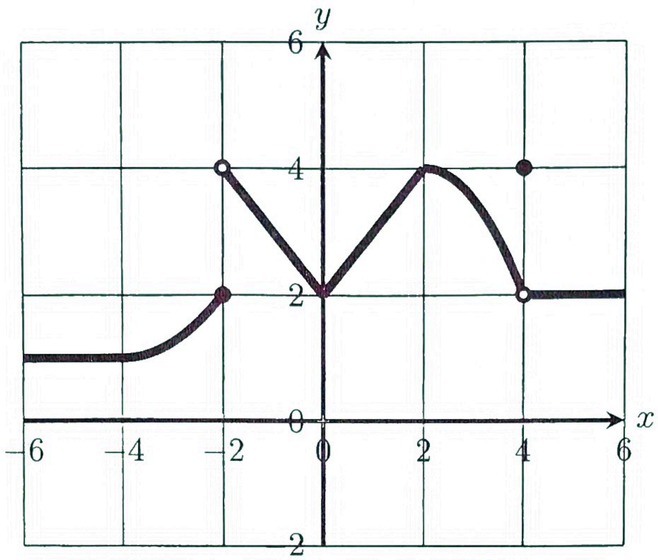
$$\lim_{x\to 3} \frac{\frac{1}{x^2} - \frac{1}{9}}{x-3}$$

$$= \lim_{x\to 3} \frac{9-x^2}{9x^2(x-3)} = \lim_{x\to 3} \frac{-(x+3)}{9x^2}$$

$$= -\frac{6}{81} = -\frac{2}{27}$$

Nicholas

3. Answer the following questions about the function $f(x)$ whose graph is depicted below. You do **not** need to justify your answers.



(a) (1 point) What is the value of $\lim_{x \to -2^+} f(x)$?

4 1 pt

(b) (1 point) What is the value of $\lim_{x \to -2^-} f(x)$?

2 1 pt

(c) (2 points) For which value(s) of a does $\lim_{x \to a} f(x)$ fail to exist? If there are none, write "none".

-2 take off 1 point if answers is 4 or any other answer

(d) (2 points) For which value(s) of a is f discontinuous at $x = a$? If there are none, write "none".

$-2, 4$ 1 point for each take off 1 point for any wrong answer

(e) (2 points) For which value(s) of a does $f'(a)$ fail to exist? If there are none, write "none".

$-2, 0, 2, 4$

take off 1 point if any missing

Katie

4. Differentiate the following functions.

(a) (4 points) $\frac{x^4}{x^3 + 5x}$

$$\frac{4x^3(x^3+5x) - x^4(3x^2+5)}{(x^3+5x)^2}$$

1 point for power rule

2 for attempting (acknowledging the quotient rule

(b) (4 points) $x^3 \cos(2x + 1)$

$$3x^2 \cos(2x+1) - 2x^3 \sin(2x+1)$$

1 point for the power rule

1 points for the product rule

1 point for the chain rule

(c) (4 points) $(\tan x - 4x^2 + 3\sqrt{x})^9$

$$9(\tan x - 4x^2 + 3\sqrt{x})^8 \left(\sec^2 x - 8x + \tfrac{3}{2}x^{-1/2}\right)$$

1 point for the chain rule

1 point for the power rule

1 point for the derivative of Tangent

(d) (4 points) $\int_3^{x^2+1} \frac{dt}{\sqrt{4 + \sin t}}$

3

$$2x\left(\frac{1}{\sqrt{4+\sin(x^2+1)}}\right)$$

1 point for recognizing that this is the fundamental thm of calculus

1 point for substituting x^2+1 for t

1 point for recognizing that they need to use the chain rule

5. (8 points) Find an equation for the line tangent to the curve $x^2y + xy^3 + 42 = 0$ at the point $(-2, 3)$.

Ryan Sasaki

$$\left.\begin{aligned}&\frac{d}{dx}\left(x^2y+xy^3+42=0\right)\\&\left(2xy+x^2y'+y^3+3xy^2y'=0\right)\end{aligned}\right\}\ 4\text{pts}$$

$$\left.\begin{aligned}&2(-2)(3)+(-2)^2y'+(3)^3+3(-2)(3)^2y'=0\\&-12+4y'+27-54y'=0\\&-50y'=-15\\&y'=\frac{3}{10}\end{aligned}\right\}\ 2\text{pts}$$

$$\left.\begin{aligned}&y=mx+b\\&y=\frac{3}{10}x+b\\&3=\frac{3}{10}(-2)+b\\&b=3+\frac{3}{5}\\&b=\frac{18}{5}\end{aligned}\right\}\ 2\text{ pts}$$

$$\boxed{y=\frac{3}{10}x+\frac{18}{5}}$$

Vasu

6. (5 points) Use linear approximation and the fact that $\sqrt[3]{8} = 2$ to estimate the value of $\sqrt[3]{8.12}$. Express your final answer as either a **fraction** in lowest terms or as a **decimal number** with two digits after the decimal point.

$f(x) = \sqrt[3]{x}$ $\quad a = 8$

1 pt

Linearization $\quad L(x) = f(a) + f'(a)(x-a)$

$$f'(x) = \frac{1}{3} x^{-2/3}$$

$$f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$$

$$L(x) = f(8) + f'(8)(x-8)$$

$$= 2 + \frac{1}{12}(x-8)$$

2 pts

$$\underbrace{\sqrt[3]{8.12} \approx 2 +}_{1\text{ pt}} \frac{1}{12}(8.12-8) = 2.01$$

OR

$\frac{801}{100}$

1pt

Thomas H.

7. Consider the equation $2x^5 + 4x^3 = 1$.

(a) (3 points) Explain why this equation must have a solution in the interval $(0, 1)$. State any theorem(s) you use in your explanation.

Mention IVT/continuity (+1)

Check $f(0)$ & $f(1)$, showing that $f(0) < 1 < f(1)$ (+1)

Reasonable Sentence (+1)

(b) (3 points) Use Rolle's theorem to explain why this equation cannot have more than one solution in the interval $(0, 1)$.

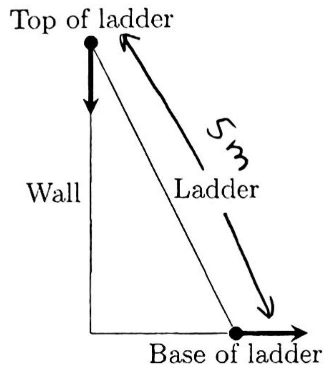
Calculate $f'(x)$ (+1)

Show $f'(x) \neq 0$ for $x \in (0,1)$. (+1)

Explanation w/ Rolle's Theorem (+1)

Pierre

8. (10 points) A ladder that is 5 meters long is leaning against a wall. Its base begins to slide horizontally away from the wall at a constant rate of 6 meters per second. Assuming that the top of the ladder slides vertically downward and always touches the wall, how quickly is the top of the ladder descending when its base is 3 meters away from the wall?



Information:

x: wall-tip base (m)

y: floor-tip. Top (m)

$\frac{dx}{dt} = 6$ m/s

Goal: $\frac{dy}{dt}\Big|_{x=3}$

Connection/ Equation $\quad 5^2 = x^2 + y^2$

Implicit derivative: $\quad 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$$\Rightarrow \; -\frac{x}{y}\frac{dx}{dt} = \frac{dy}{dt}.$$

Replace: $\quad \frac{dy}{dt}\Big|_{x=3} = -\frac{3}{y}(6)$

$$= -\frac{3}{4}\cdot 6 \qquad \left(y = \sqrt{25-9} = 4\right)$$

$$= \boxed{-\frac{9}{2}\ \text{m/s}}$$

- Information + Goal: 2pts.
- Equation: 2pts.
- Derivation: 3pts.
- Plugin: 2pts
- Answer: 1pt.

Sydney Fields

9. (10 points) A company that sells canned soup wants to design a cylindrical can that encloses as much soup as possible using $24\pi\,\text{cm}^2$ of metal. What is the maximum volume that such a cylinder (i.e., a cylinder whose surface area is $24\pi\,\text{cm}^2$) can enclose? (Recall that the volume of a cylinder of radius r and height h is $\pi r^2 h$, and the surface area is $2\pi rh + 2\pi r^2$.)

Maximize volume. $V = \pi r^2 h$ $\quad \frac{dV}{dt} = ?$

$2\pi rh + 2\pi r^2 = 24\pi$

$2\pi r(h+r) = 24\pi$

$h + r = \frac{12}{r}$

$h = \frac{12}{r} - r$

$V = \pi r^2\left(\frac{12}{r} - r\right)$

$V = \pi(12r - r^3)$

$\frac{dV}{dt} = \pi(12 - 3r^2)$

$0 = \pi(12 - 3r^2)$

$0 = 12 - 3r^2$

$-12 = -3r^2$

$4 = r^2$

$r = \pm 2$

$r = 2$

$-$ $\quad +$ max $\quad -$

$-2 \quad 2$

$r = 2,\ h = \frac{12}{2} - 2 = 4$

$V = \pi(2)^2 \cdot 4$

$\boxed{= 16\pi\ \text{cm}^3}$

Rubric

+1 - if they list assumptions correctly

+2 - solve for h

+5 - differentiate correctly

+1 - if they get radius

+1 - if they say volume

(if everything is wrong because they didn't write 24π I took off 1 point)

if everything wrong b/c didn't write 24π & still wrong, but right idea, only 1 pt.)

For all parts: -1 if extra answers included

10. Consider the function $f(x) = \dfrac{x+1}{2x+3}$. The first and second derivatives of f are $f'(x) = \dfrac{1}{(2x+3)^2}$ and $f''(x) = \dfrac{-4}{(2x+3)^3}$, respectively. Answer the following questions about f.

-1 if ~~the~~ the answer is almost right but missing something meaningful

(Note: This problem continues on the next page and the page thereafter.)

HUGH CHOU

(a) (2 points) On which open interval(s) is f increasing? If there are none, write "none".

Full credit: $(-\infty, -\frac{3}{2}) \cup (-\frac{3}{2}, \infty)$ ~~or~~ $x < -\frac{3}{2}$, $x > -\frac{3}{2}$

Partial (1 pt): $f'(x) > 0$, $-\frac{3}{2}$ turns up somewhere as an endpoint

either single interval $(-\infty, \infty)$ if they have a $-\frac{3}{2}$

$(-\infty, a) \cup (a, \infty)$ if a is a reasonable arithmetic error

(b) (2 points) On which open interval(s) is f decreasing? If there are none, write "none".

Full credit: none

partial (1 pt): $f'(x) < 0$

(c) (2 points) On which open interval(s) is f concave up? If there are none, write "none".

Full credit: $(-\infty, -\frac{3}{2})$, $x < -\frac{3}{2}$

Partial (1 pt): $f''(x) > 0$

$(-\infty, a)$ if a is a reasonable arithmetic error

$-\frac{3}{2}$ turns up as an endpoint in a reasonable way

(*Problem 10, continued. Recall that* $f(x) = \frac{x+1}{2x+3}$, $f'(x) = \frac{1}{(2x+3)^2}$, $f''(x) = \frac{-4}{(2x+3)^3}$.)

(d) (2 points) On which open interval(s) is f concave down? If there are none, write "none".

Full credit: $(-\frac{3}{2}, \infty)$, $x > -\frac{3}{2}$

Partial (1pt): $f''(x) < 0$

(a, ∞) if a is a reasonable arithmetic error

$-\frac{3}{2}$ as an endpoint in a reasonable way

(e) (2 points) Find the vertical asymptote(s) in the graph of f. If there are none, write "none".

Full credit: $-\frac{3}{2}$, $x = -\frac{3}{2}$

Partial ~~credit~~ (1pt): $f(x)$ is undefined

$\lim_{x \to c^{\pm}} f(x) = \infty$

with wrong c

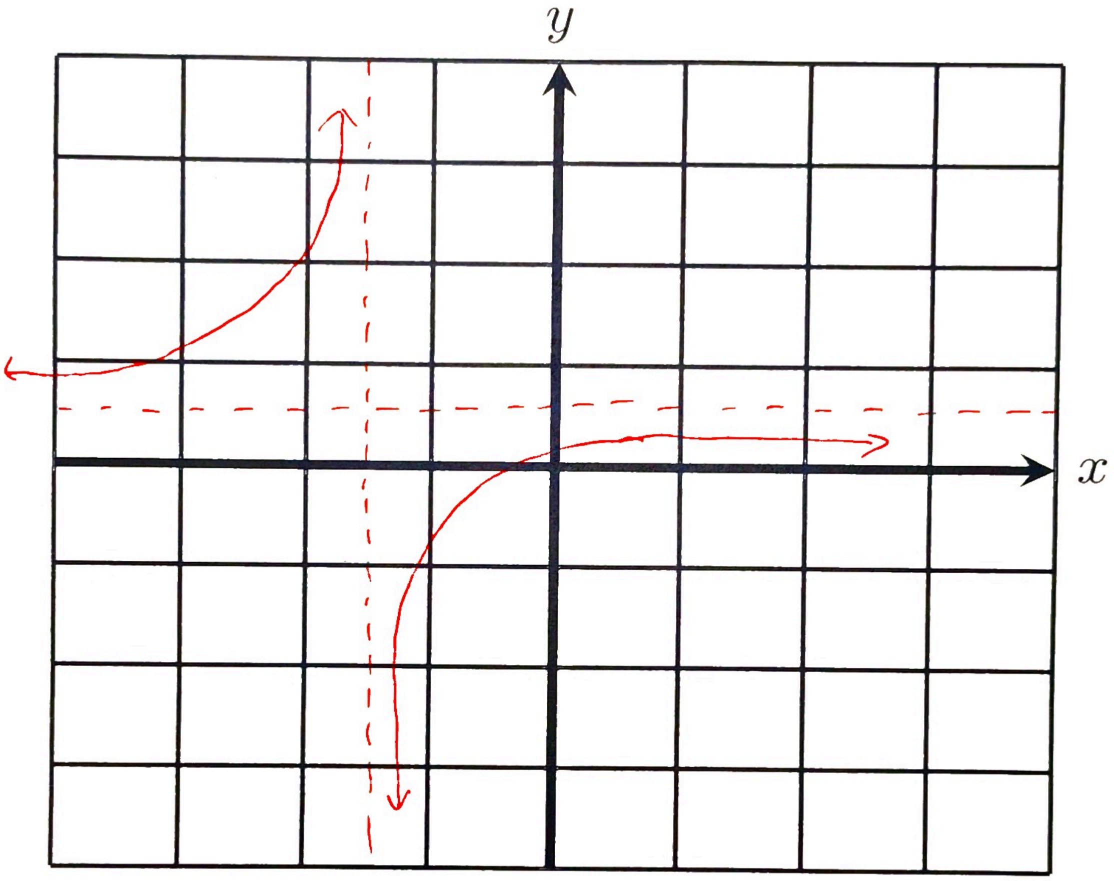
(f) (2 points) Find the horizontal asymptote(s) in the graph of f. If there are none, write "none".

Full credit: $y = \frac{1}{2}$, $\frac{1}{2}$

Partial (1pt): $\lim_{x \to \pm\infty} f(x)$

(*Problem 10, continued. Recall that* $f(x) = \dfrac{x+1}{2x+3}$, $f'(x) = \dfrac{1}{(2x+3)^2}$, $f''(x) = \dfrac{-4}{(2x+3)^3}$.)

(g) (3 points) Sketch the graph of f on the interval $-4 \le x \le 4$.



full credit ↑

partial (2pts): Graph looks exactly like what parts a–f describe it as

partial (1 pt): Graph looks a bit like what's described

11. Calculate the following integrals. Be sure to simplify your answers.

(a) (5 points) $\int \frac{(x-\sqrt{x})^2}{x}\,dx$

$$= \int \frac{x^2 - 2x\sqrt{x} + x}{x}\,dx \quad 1$$

$$= \int x - 2\sqrt{x} + 1 \; dx \quad 1$$

for $+C$

$$= \frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + x + C \quad 1 + 1 \quad = \frac{x^2}{2} - \frac{4x^{\frac{3}{2}}}{3} + x + C, \quad 1$$

where C is an arbitrary constant.

(b) (5 points) $\int_0^{\sqrt{\pi}/2} x\cos(x^2)\,dx$

1 Put $u = x^2$. Then $du = 2x\,dx$. 1 When $x=0$, $u=0$, and $\frac{1}{2}$ when $x = \frac{\sqrt{\pi}}{2}$, $u = \frac{\pi}{4}$. $\frac{1}{2}$

$$\text{So } \int_0^{\frac{\sqrt{\pi}}{2}} x\cos(x^2)\,dx = \frac{1}{2}\int_0^{\frac{\sqrt{\pi}}{2}} 2x\cos(x^2)\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos(u)\,du \quad \frac{1}{2} \quad = \frac{1}{2}\sin(u)\Big|_0^{\frac{\pi}{4}} \quad 1$$

$$= \frac{1}{2}\left(\sin\left(\frac{\pi}{4}\right) - \sin(0)\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1}{\sqrt{2}} - 0\right) = \frac{1}{2\sqrt{2}}. \quad \frac{1}{2}$$

(c) (5 points) $\int \frac{2x+3}{(x^2+3x+4)^2}\,dx$

Put $x^2 + 3x + 4 = u$. 1 Then $du = (2x+3)\,dx$. 1 So,

$$\int \frac{2x+3}{(x^2+3x+4)^2}\,dx = \int \frac{du}{u^2} \quad 1 \quad = -\frac{1}{u} + C \quad 1 \quad 1 \text{ for } +C$$

$$= -\frac{1}{x^2+3x+4} + C,$$

where C is an arbitrary constant.

Nicholas

12. A sailboat is traveling across the Pacific ocean but the winds are shifting unfavorably, so the boat is accelerating at a rate of -2 kilometers/hour^2. Suppose that at time $t = 0$, the boat is 300 kilometers east of Honolulu and traveling east at a speed of 10 kilometers/hour.

(a) (5 points) Write down a function that describes the position of the boat (in kilometers east of Honolulu) at time t (in hours).

1 pt $\begin{cases} v(0) = 10 \\ s(0) = 300 \text{ east} \\ \frac{dv}{dt} = -2 \end{cases}$

$v(t) = -2t + C$ 1 pt

$\Rightarrow v(t) = -2t + 10$ 1 pt

1 pt $\begin{cases} \frac{ds}{dt} = -2t + 10 \\ s(t) = -t^2 + 10t + C \end{cases}$

$\Rightarrow$ At $t = 0$, $s(0) = 300$

$\Rightarrow s(t) = -t^2 + 10t + 300$ 1 pt

(b) (3 points) At what time t (in hours) will the boat reach an (eastward) speed of 4 kilometers/hour?

$-2t + 10 = 4$

$\Rightarrow t = 3$ hrs

3 pts versus 0

if calculation error

-1 pt

(c) (4 points) At what time t (in hours) will the boat first reach a position of 324 kilometers east of Honolulu?

$-t^2 + 10t + 300 = 324$ ← 1 pt

$\Rightarrow t^2 - 10t + 24 = 0$ 1 pt

$\Rightarrow \cancel{(t-12)}$ $(t-6)(t-4) = 0$ 1 pt

$\Rightarrow t = 4$ 1 pt.

Vasu ·

13. Consider the function $f(x) = 1 + x^2$.

(a) (4 points) Estimate $\int_0^3 f(x)\,dx$ with a Riemann sum using 3 subintervals of equal width and left endpoints.

1

~~Riemann sum =~~

Width of each subinterval $= \frac{3-0}{3} = 1$

2

Riemann sum $= f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

$= (1+0^2)\cdot 1 + (1+1^2)\cdot 1 + (1+2^2)$

$= 1 + 2 + 5 = 8 \leftarrow$ 1

-2 if right endpts used

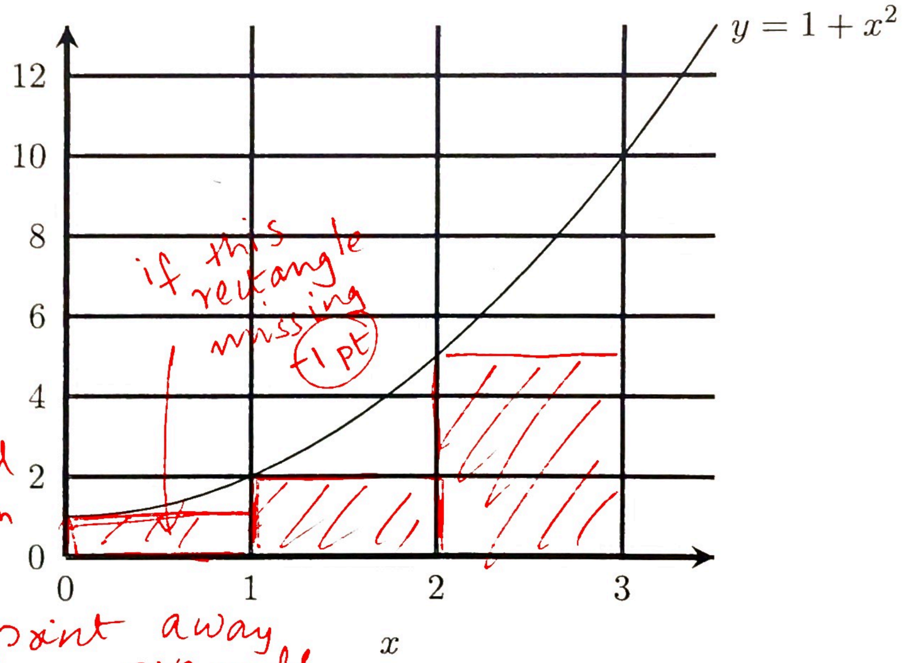
(b) (4 points) In the plot below, sketch the rectangles that you used in part (a).

4 pts for correct

otherwise use your discretion

If right endpoints used and calculation above matches

— take 1 point away overall



(c) (2 points) Which is larger: your estimate from part (a) or the exact value of $\int_0^3 f(x)\,dx$?

2 pts for correct answer

0 pts if wrong

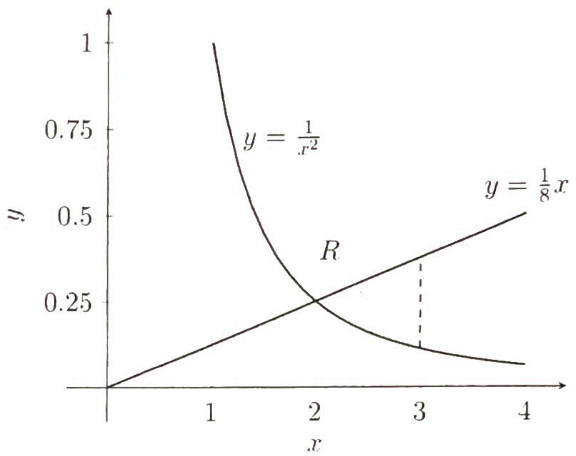
Actual area > estimate

~~[illegible] if [illegible]~~ missing

if right endpoints used & correct conclusion ⇒ 2 pts

Thomas H

14. Let R be the region bounded by the curve $y = \frac{1}{x^2}$ and the lines $x = 3$ and $y = \frac{1}{8}x$, as shown below.



(a) (5 points) Find the area of the region R.

bounds (+1)

Integral

$$\int_2^3 \underbrace{\frac{1}{8}x - \frac{1}{x^2}}_{\text{integrand (+2)}} \, dx$$

+3

Integrands & Eval

+2

$$\frac{1}{16}x^2 + x^{-1} \Big|_2^3$$

+1

Eval

$$\boxed{\frac{9}{16} + \frac{1}{3} - \frac{4}{16} - \frac{1}{2}}$$

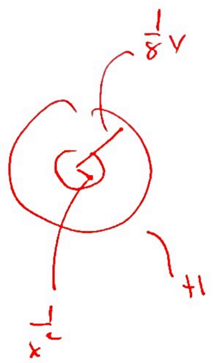
$$\frac{5}{16} + \frac{1}{3} - \frac{1}{2} =$$

Thomas H

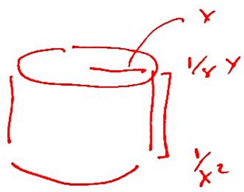
(b) (4 points) Set up **but do not evaluate** an integral for the volume of the solid obtained by rotating R about the x-axis.

$$V = \int_2^3 \underbrace{\pi\left(\tfrac{1}{8}x\right)^2 - \pi\left(\tfrac{1}{x^2}\right)^2}_{(+2)} \, dx$$

+1



(c) (4 points) Set up **but do not evaluate** an integral for the volume of the solid obtained by rotating R about the y-axis.



$$\int_2^3 \underbrace{2\pi x}_{+1} \underbrace{\left(\tfrac{1}{8}x - \tfrac{1}{x^2}\right)}_{+1} dx$$

+1

+1

or

Two Integrals