MATH 241 COMMON FINAL EXAM, SPRING 2022

You have 120 minutes.

No books, no notes, no electronic devices.

YOU MUST SHOW ALL WORK. NO NEED TO SIMPLIFY ANSWERS.

Name ______________________

Instructor Name ______________________

Section Number ______________________

Grade table (for instructor's use only)

1. (16pts) ________
2. (4pts) ________
3. (8pts) ________
4. (20pts) ________
5. (5pts) ________
6. (8pts) ________
7. (8pts) ________
8. (10pts) ________
9. (24pts) ________
10. (15pts) ________
11. (8pts) ________
12. (10pts) ________
13. (14pts) ________

Total Score ____________ (/150 points)

1. Calculate the following limits. **Do not** use L'Hospital's rule. If the limit is infinite, specify whether it is ∞ or $-\infty$.

   (a) (4pts) $\displaystyle\lim_{x\to -2^-} \frac{x+2}{x^2+4x+4}$

   (b) (4pts) $\displaystyle\lim_{x\to 0^+} \sqrt{x^2+x}\sin(x)$

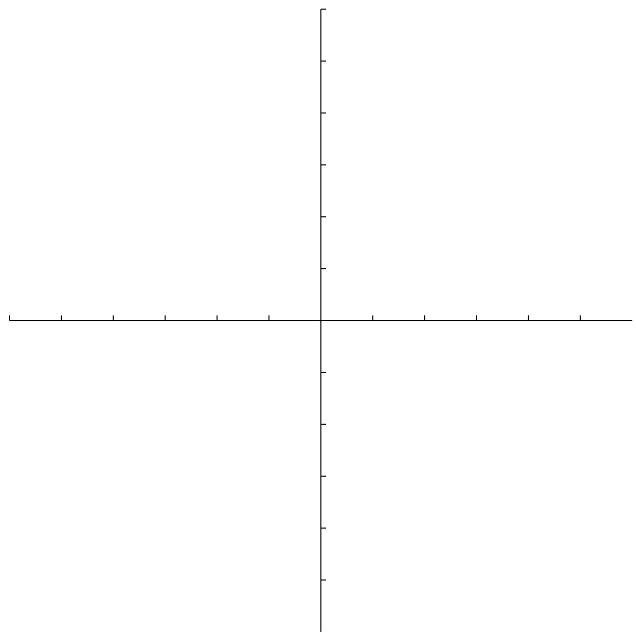
   (c) (4pts) $\displaystyle\lim_{x\to -2} \frac{2-|x|}{2+x}$

   (d) (4pts) $\displaystyle\lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right)$

2. Consider the function f defined by

$$f(x) = \begin{cases} -x & \text{if } x < -1 \\ \sin\left(\frac{\pi}{2}x\right) & \text{if } -1 \leq x < 1 \\ 2 & \text{if } x = 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

(a) (2pts) Sketch the graph of f.



(b) (1pt) Find the values a such that $\lim_{x \to a} f(x)$ does not exist. No justification needed.

(c) (1pt) Find the values a such that $f(x)$ is discontinuous at $x = a$. No justification needed.

3. Consider the function $f(x) = \frac{1}{\sqrt{x}}$.

(a) (4pts) Using the definition of the derivative as a limit, compute $f'(x)$.
(Warning: you will not get credit if you use the rules of differentiation.)

(b) (2pts) What is the domain of f'?

(c) (2pts) Find the equation of the tangent line to the curve $y = f(x)$ at the point $(4, 1/2)$.

4. In each of the following, calculate the derivative $\frac{dy}{dx}$. You do not need to simplify your answers.

(a) (5pts) $y = \dfrac{x^2 + 2}{x^5 + 3}$

(b) (5pts) $y = \sqrt{x} \cos(x^2)$

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In each of the following, calculate the derivative $\frac{dy}{dx}$. You do not need to simplify your answers.

(c) (5pts) $y = \left(\sqrt{x} + \frac{2}{x}\right)^7$

(d) (5pts) $y = \int_0^{1/x} \sin^4 t \, dt$

5. (5pts) Find the slope of the tangent line to the graph of $x^3 - 3x^2y + 2xy^2 = 0$ at the point $(1, 1)$.

6. Consider the equation $2x - 1 = \sin x$.

   (a) (4pts) Explain why the equation has a solution in the interval $[0, \pi/2]$. You may use the Intermediate Value Theorem.

   (b) (4pts) Explain why the equation cannot have more than one solution in the interval $[0, \pi/2]$. You may use Rolle's Theorem or the Mean Value Theorem.

7. (8pts) A spherical snowball melts so that its surface area decreases at a rate of $1\,\text{cm}^2$ per minute. Find the rate at which the diameter decreases when the radius is $5\,\text{cm}$.

   (Recall that the surface area of a sphere of radius r is $4\pi r^2$.)

8. (10pts) A box with a square base and open top must have a volume of $32\,\mathrm{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

9. Let $f(x) = \dfrac{1}{x^2-1}$. Then $f'(x) = -\dfrac{2x}{(x^2-1)^2}$ and $f''(x) = \dfrac{6x^2+2}{(x^2-1)^3}$ (you may take those formulas for granted).

(a) (2pts) Find the domain of f.

(b) (2pts) Find the intercepts with the x and y-axes, if there are any.

(c) (2pts) Find the vertical asymptotes of f, if there are any.

(d) (2pts) Find the horizontal asymptotes of f, if there are any.

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(e) (4pts) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(f) (4pts) Find the local minimum values and the local maximum values, if there are any.

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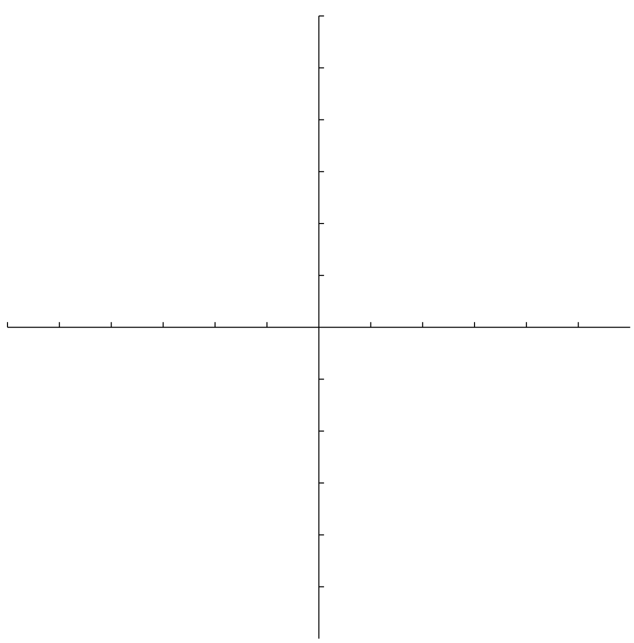
(g) (4pts) Find the intervals on which f is concave up, and the intervals on which f is concave down.

(h) (2pts) Identify all inflection points, if there are any.

(i) (2pts) Sketch the graph of f.



10. Evaluate the following integrals.

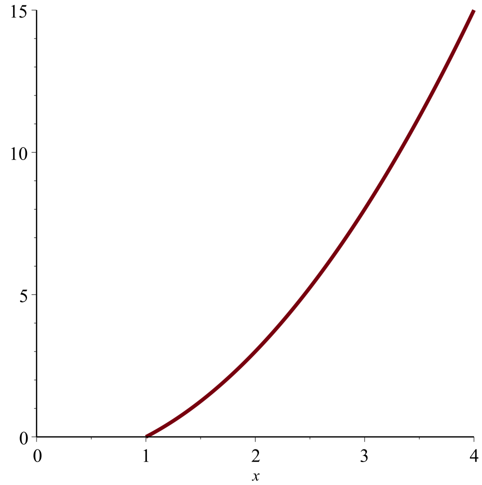
(a) (5pts) $\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$

(b) (5pts) $\int \frac{x^3}{(x^4-5)^2}\, dx$

(c) (5pts) $\int_0^2 |x-1|\, dx$

11. (8pts) A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6\,\mathrm{cm}\,/\,\mathrm{s}$ and its initial position is $9\,\mathrm{cm}$ in the positive direction from the origin. Find the position of the particle after 2 seconds.

12. Consider the parabola $y = x^2 - 1$ between $x = 1$ and $x = 4$, pictured below.

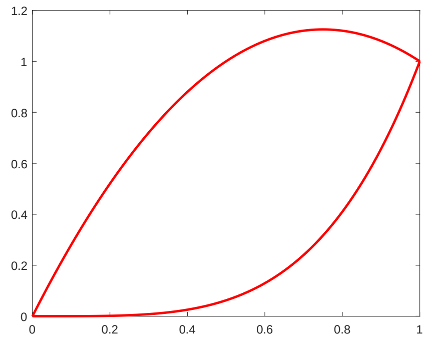


(a) (6pts) Estimate the area under the parabola and above the x-axis between $x = 1$ and $x = 4$ with a Riemann sum, using three subintervals of equal width and right endpoints.

(b) (2pts) Sketch the rectangles that you used in part (a) on the provided graph.

(c) (2pts) Is your answer in (a) larger or smaller than the true area $\int_1^4 (x^2 - 1)\, dx$? Explain.

13. Consider the region R in the first quadrant bounded by the curves $y = x^4$ and $y = 3x - 2x^2$, pictured below. The two curves intersect at the points $(0, 0)$ and $(1, 1)$.



(a) (6pts) Find the area of the region R.

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(b) (4pts) Set up **but do not evaluate** an integral for the volume of the solid obtained by rotating R about the x-axis.

(c) (4pts) Set up **but do not evaluate** an integral for the volume of the solid obtained by rotating R about the y-axis.