1. Calculate the following limits. **Do not** use L'Hospital's rule. If the limit is infinite, specify whether it is ∞ or $-\infty$.

(a) (4pts) $\lim_{x\to -2^-} \frac{x+2}{x^2+4x+4}$

$$\lim_{x\to -2^-} \frac{x+2}{x^2+4x+4} = \lim_{x\to -2^-} \frac{x+2}{(x+2)^2} \quad \text{2pts.}$$

$$= \lim_{x\to -2^-} \frac{1}{x+2} \quad \text{1pt.}$$

$$= \frac{1}{0^-} = \boxed{-\infty} \quad \text{1pt.}$$

(b) (4pts) $\lim_{x\to 0^+} \sqrt{x^2+x}\sin(x)$

By the substitution rule 2pts.

$$\lim_{x\to 0^+} \sqrt{x^2+x}\,\sin x = \sqrt{0^2+0}\,\sin(0) = 0 \quad \text{1pt.} \quad \text{1pt.}$$

(c) (4pts) $\lim_{x\to -2} \frac{2-|x|}{2+x}$

Since x will approach -2 $\Rightarrow$ $|x| = -x$ $(x<0)$. 1pt.

$$\Rightarrow \lim_{x\to -2} \frac{2-|x|}{2+x} = \lim_{x\to -2} \frac{2+x}{2+x} = \lim_{x\to -2} 1 = \boxed{1}$$

2pts. 1pt.

(d) (4pts) $\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right)$

$$\sqrt{9x^2+x} - 3x = \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} = \frac{x}{x\left(\sqrt{9+1/x}+3\right)} \quad \text{2pts.}$$

$$= \frac{1}{\sqrt{9+1/x}+3}$$

So $\lim_{x\to\infty} \frac{1}{\sqrt{9+1/x}+3} = \boxed{\frac{1}{6}}$ 1pt

1pt.

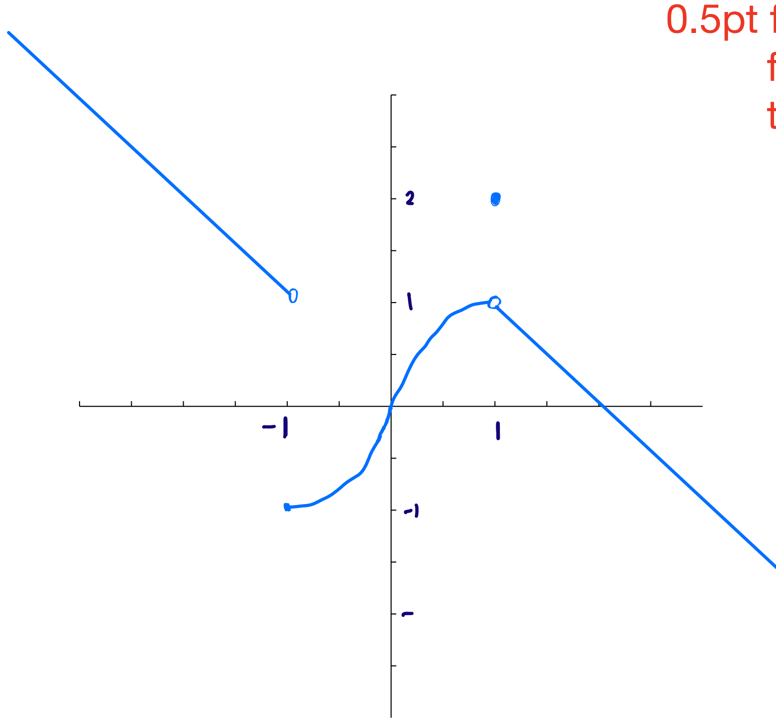
substitution rule.

2. Consider the function f defined by

$$f(x) = \begin{cases} -x & \text{if } x < -1 \\ \sin\left(\frac{\pi}{2}x\right) & \text{if } -1 \le x < 1 \\ 2 & \text{if } x = 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

(a) (2pts) Sketch the graph of f.

0.5pt for each of the four sections of the graph



(b) (1pt) Find the values a such that $\lim_{x \to a} f(x)$ does not exist. No justification needed.

$a = -1$ $\longrightarrow$ 1 pt

(c) (1pt) Find the values a such that $f(x)$ is discontinuous at $x = a$. No justification needed.

$a = -1$ $\longrightarrow$ 0.5pt

$a = 1$ $\longrightarrow$ 0.5pt

3. Consider the function $f(x) = \frac{1}{\sqrt{x}}$.

(a) (4pts) Using the definition of the derivative as a limit, compute $f'(x)$.
(Warning: you will not get credit if you use the rules of differentiation.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \frac{\sqrt{x+h}\cdot\sqrt{x}}{\sqrt{x+h}\,\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x+h}\,\sqrt{x})} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h(\sqrt{x+h}\,\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-h}{h(\sqrt{x+h}\,\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\,\sqrt{x}\,(\sqrt{x}+\sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x}\,\sqrt{x}\,(\sqrt{x}+\sqrt{x})}$$

$$= \frac{-1}{x(2\sqrt{x})} = -\frac{1}{2x\sqrt{x}}$$

or $-\frac{1}{2}x^{-3/2}$

or $\frac{1}{2x^{3/2}}$

-4 for using power rule.

-3 incorrect difference quotient; vary depending on how incorrect.

-1/2 per small algebraic/arithmetic error
-1 per big arithmetic/algebraic error

} assuming work after is consistent following mistake

(b) (2pts) What is the domain of f'?

-1 including 0

-2 including $(-\infty, 0)$

$\boxed{D = (0, \infty)}$

(c) (2pts) Find the equation of the tangent line to the curve $y = f(x)$ at the point $(4, 1/2)$.

$$f'(x) = -\frac{1}{2x^{3/2}}$$

$$f'(4) = -\frac{1}{2\cdot 4^{3/2}}$$

$$= -\frac{1}{2\cdot 8}$$

$$= -\frac{1}{16}$$

$$\boxed{y - \tfrac{1}{2} = -\tfrac{1}{16}(x-4) \quad \text{or} \quad y = -\tfrac{1}{16}x + \tfrac{3}{4}}$$

-1/2 small error in slope-intercept or point-slope form.

-1 reversing roles of $x_1 = 4$ & $y_1 = 1/2$.

-1 incorrect evaluation of $f'(4)$

4. In each of the following, calculate the derivative $\frac{dy}{dx}$. You do not need to simplify your answers.

(a) (5pts) $y = \dfrac{x^2+2}{x^5+3}$

$$\therefore \frac{dy}{dx} = \frac{\overbrace{(x^5+3)\cdot\frac{d}{dx}\left[x^2+2\right]}^{1\text{ pt}} - \overbrace{(x^2+2)\frac{d}{dx}\left[x^5+3\right]}^{1\text{ pt}}}{(x^5+3)^2 \leftarrow 1\text{ pt}}$$

$$= \boxed{\frac{\overbrace{2x}^{1\text{ pt}}(x^5+3) - \overbrace{5x^4}^{1\text{ pt}}(x^2+2)}{(x^5+3)^2}}$$

(b) (5pts) $y = \sqrt{x}\cos(x^2)$

$$\therefore \frac{dy}{dx} = \overbrace{\frac{d}{dx}\left[\sqrt{x}\right]\cdot\cos(x^2)}^{1\text{ pt}} + \overbrace{\sqrt{x}\,\frac{d}{dx}\left[\cos(x^2)\right]}^{1\text{ pt}}$$

$$= \boxed{\underbrace{\left(\tfrac{1}{2}x^{-1/2}\right)}_{1\text{ pt}}\cos(x^2) + \sqrt{x}\underbrace{\left[-2x\sin(x^2)\right]}_{2\text{pt}}}$$

2pt – 1 pt correct answer
1pt work/steps

Problem continues on the next page...

In each of the following, calculate the derivative $\frac{dy}{dx}$. You do not need to simplify your answers.

(c) (5pts) $y = \left(\sqrt{x} + \frac{2}{x}\right)^7$

Each line 1 pt

$u = \sqrt{x} + \frac{2}{x} \Rightarrow y = u^7$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} - 2x^{-2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 7u^6 \cdot \left(\frac{1}{2}x^{-1/2} - 2x^{-2}\right)$

$= \boxed{7\left(\sqrt{x} + \frac{2}{x}\right)^6 \left[\frac{1}{2}x^{-1/2} - 2x^{-2}\right]}$

(d) (5pts) $y = \int_0^{1/x} \sin^4 t \, dt$

Each line 1 pt

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\int_0^{1/x} \sin^4 t \, dt\right]$

$u = \frac{1}{x} \Rightarrow \frac{du}{dx} = -\frac{1}{x^2} = -x^{-2}$

$\therefore \frac{dy}{dx} = \frac{d}{du}\left[\int_0^{u} \sin^4 t \, dt\right] \cdot \frac{du}{dx}$

$= \sin^4 u \times -\frac{1}{x^2}$

$= \boxed{-\frac{\sin^4\left(\frac{1}{x}\right)}{x^2}}$

5. (5pts) Find the slope of the tangent line to the graph of $x^3 - 3x^2y + 2xy^2 = 0$ at the point $(1, 1)$.

Using Implicit differentiation:

2 pts for implicit diff.

$$3x^2 - 6xy - 3x^2\frac{dy}{dx} + 2y^2 + 2x \cdot 2y\frac{dy}{dx} = 0$$

2 pts

$$\Rightarrow (-3x^2 + 4xy)\frac{dy}{dx} = -3x^2 + 6xy - 2y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3x^2 + 6xy - 2y^2}{-3x^2 + 4xy}$$

1 pts

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-3+6-2}{-3+4} = \frac{1}{1} = 1$$

So, the slope of the tangent line is 1.

* Sentences like "the graph should look like this" are not acceptable without explanations

* No explanation = no points

6. Consider the equation $2x - 1 = \sin x$.

(a) (4pts) Explain why the equation has a solution in the interval $[0, \pi/2]$. You may use the Intermediate Value Theorem.

$f(x) = 2x - 1 - \sin(x)$

$f(0) = -1 < 0$ — +1 pt

$f(\pi/2) = \pi - 2 > 0$ — +1 pt

$f(x)$ is continuous — +1 pt

Acceptable logic arguments — +1 pt

↳ Using IVT

↳ $f(x)$ cannot jump over 0

↳ drawing a continuous curve using the above + words

Don't be picky on words

(b) (4pts) Explain why the equation cannot have more than one solution in the interval $[0, \pi/2]$. You may use Rolle's Theorem or the Mean Value Theorem.

Method by contradiction

+1 pt — $f'(x) = 2 - \cos(x)$

+1 pt — $f'(x)$ is never = to 0

+2 pts — Contradicts Rolle's theorem

Example

Let a and b be two solutions such that $a \neq b$ and $f(a) = f(b) = 0$. It satisfies Rolle's theorem but $f'(x)$ cannot be zero hence there is a contradiction. Therefore, the equation cannot have more than one solution.

Method using concepts

$f'(x) = 2 - \cos(x)$ — +1 pt

$f(x)$ is always increasing — +1 pt

$f(x)$ is continuous — see (a)

Acceptable logic arguments — +2 pts

↳ Since $f(x)$ is continuous and always increasing, $f(x)$ cannot cross $y=0$ more than once. Therefore, the equation cannot have more than one solution.

Don't be picky on words

7. (8pts) A spherical snowball melts so that its surface area decreases at a rate of $1\,\mathrm{cm}^2$ per minute. Find the rate at which the diameter decreases when the radius is $5\,\mathrm{cm}$.

(Recall that the surface area of a sphere of radius r is $4\pi r^2$.)

$A = 4\pi r^2$ (1 pt)

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$ (3 pts)

$$1 = 8\pi(5)\frac{dr}{dt}$$ (1 pt)

$$\frac{1}{40\pi} = \frac{dr}{dt}$$ (1 pt)

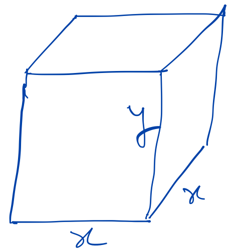
☆ Ignore ± signs.

☆ Ignore the units.

$$\left.\begin{aligned} d &= 2r \\ \frac{dd}{dt} &= 2\frac{dr}{dt} \end{aligned}\right\}$$ (1 pt)

$$\left.\begin{aligned} \frac{dd}{dt} &= 2\left(\frac{1}{40\pi}\right) \\ &= \frac{1}{20\pi} \end{aligned}\right\}$$ (1 pt)

8. (10pts) A box with a square base and open top must have a volume of $32\,\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.



Constraint is volume, $x^2y = 32 \Rightarrow y = \dfrac{32}{x^2}$ } 2 pts

3 pts. { Material needs to construct

surface Area: $A = x^2 + 4xy$

$= x^2 + 4 \cdot x \cdot \dfrac{32}{x^2}$

$= x^2 + \dfrac{128}{x}$; $x > 0$

2 pts { $\dfrac{dA}{dx} = 2x - \dfrac{128}{x^2}$

Critical point, $2x - \dfrac{128}{x^2} = 0 \Rightarrow 2x^3 = 128 \Rightarrow x^3 = 64 \Rightarrow x = 4$

2 pts { Second derivate test.

Same points will be applicable for first derivative test.

$\dfrac{d^2A}{dx^2} = 2 + \dfrac{256}{x^3}$

at $x = 4$; $A'' > 0$, So $x = 4$ is a minimum point.

1 pts. { So, For minimizing materials $x = 4$

and $y = \dfrac{32}{16} = 2$

9. Let $f(x) = \frac{1}{x^2-1}$. Then $f'(x) = -\frac{2x}{(x^2-1)^2}$ and $f''(x) = \frac{6x^2+2}{(x^2-1)^3}$ (you may take those formulas for granted).

(a) (2pts) Find the domain of f.

$$(-\infty,-1) \cup (-1,1) \cup (1,\infty)$$

1 point per discontinuity

(b) (2pts) Find the intercepts with the x and y-axes, if there are any.

$x=0 \rightarrow f(0)=-1$

$y=0 \rightarrow f(\pm\infty)=0 \leftarrow$ (optional)

(c) (2pts) Find the vertical asymptotes of f, if there are any.

$x=1$

$x=-1$

1 point per asymptote

(d) (2pts) Find the horizontal asymptotes of f, if there are any.

$$\lim_{x\to\pm\infty}\left(f(x)\right)=0$$

1 point method

$y=0$

1 point result

Problem continues on the next page...

(e) (4pts) Find the intervals on which f is increasing and the intervals on which f is decreasing.

1pt Critical points $x = -1, 0, 1$

1pt method (Table, etc)

1pt Increasing: $(-\infty, -1) \cup (-1, 0)$

1pt Decreasing: $(0, 1) \cup (1, \infty)$

(f) (4pts) Find the local minimum values and the local maximum values, if there are any.

maximum: $x = 0$ $f(0) = -1$

minimum: None ← (optional)

Problem continues on the next page...

(g) (4pts) Find the intervals on which f is concave up, and the intervals on which f is concave down.

2pts Concave up: $(-\infty, -1) \cup (1, \infty)$

2pts Concave down: $(-1, 1)$

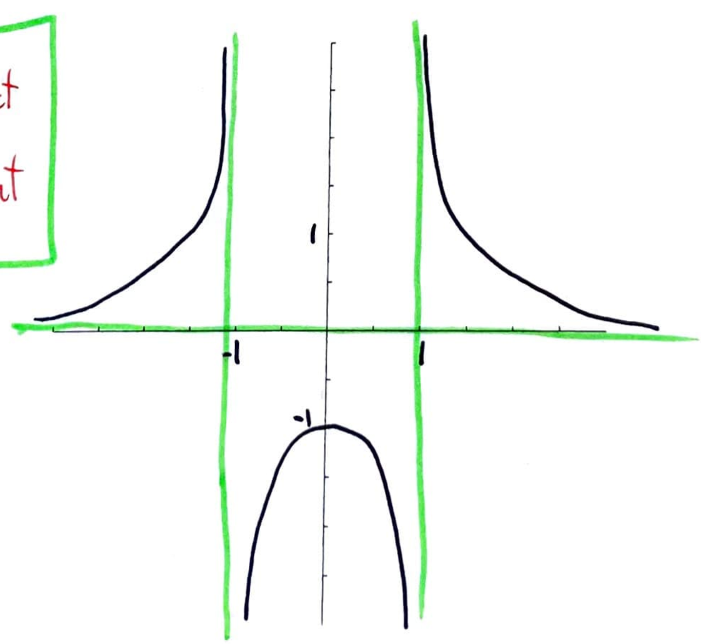
1pt for method if calculations are wrong

(h) (2pts) Identify all inflection points, if there are any.

None

(i) (2pts) Sketch the graph of f.

2pts if perfect
1pt if coherent



10. Evaluate the following integrals.

(a) (5pts) $\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$

$u = x^2$ (1) when $x = 0$ $u = 0$ (1)

$du = 2x\, dx$ (1) $x = \sqrt{\pi}$ $u = \pi$

$I = \frac{1}{2}\int_0^{\pi} \sin u\, du$ (1) $= \frac{1}{2}\left[-\cos u\right]_0^{\pi} = \frac{1}{2}(1+1) = 1$ (1)

(b) (5pts) $\int \frac{x^3}{(x^4-5)^2}\, dx$

$I = \int x^3 (x^4-5)^{-2}\, dx$ (1)

$u = x^4 - 5$ (1)

$du = 4x^3\, dx$ (1)

$I = \frac{1}{4}\int u^{-2}\, du$ (1) $= \frac{1}{4}\frac{u^{-1}}{-1} + C = -\frac{1}{4u} + C = -\frac{1}{4(x^4-5)} + C$ (1)

(c) (5pts) $\int_0^2 |x-1|\, dx$

$I = \int_0^1 -(x-1)\,dx + \int_1^2 (x-1)\, dx$ (3)

$= \left[-\frac{x^2}{2} + x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$ (1)

$= \left(\frac{1}{2} - 0\right) + \left(0 - \left(-\frac{1}{2}\right)\right) = 1$ (1)

11. (8pts) A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6\,\mathrm{cm}/\mathrm{s}$ and its initial position is $9\,\mathrm{cm}$ in the positive direction from the origin. Find the position of the particle after 2 seconds.

Stating the problem (optional)

1/8 $a(t) = 6t + 4 \qquad v(0) = -6 \qquad x(0) = 9 \qquad x(2) = ?$

Finding the velocity over time

2/8 $v(t) = \int a(t)\,dt + v_0$ relation $a = \frac{dv}{dt}$

3/8 $= 3t^2 + 4t + v_0$ integration

4/8 $\begin{cases} v(0) = -6 = v_0 \\ v(t) = 3t^2 + 4t - 6 \end{cases}$ v_0

Finding the position over time

5/8 $x(t) = \int v(t)\,dt + x_0$ relation $v = \frac{dx}{dt}$

6/8 $= t^3 + 2t^2 - 6t + x_0$ integration

7/8 $\begin{cases} x(0) = 9 = x_0 \\ x(t) = t^3 + 2t^2 - 6t + 9 \end{cases}$ x_0

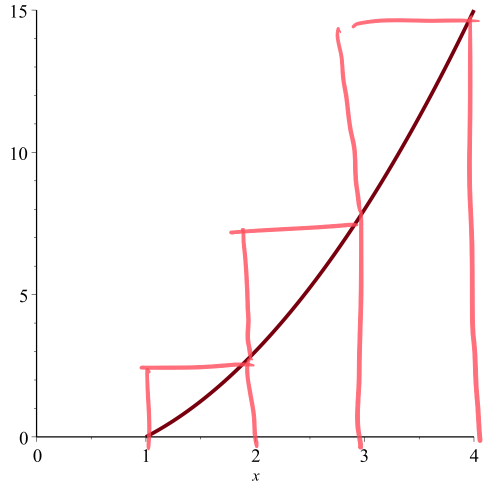
Evaluating the answer

8/8 $x(2) = 13\ \mathrm{cm}$ Full answer

— Order may vary

— Units may be dropped

12. Consider the parabola $y = x^2 - 1$ between $x = 1$ and $x = 4$, pictured below.



(a) (6pts) Estimate the area under the parabola and above the x-axis between $x = 1$ and $x = 4$ with a Riemann sum, using three subintervals of equal width and right endpoints.

$\Delta x = \frac{4-1}{3} = 1$

$R_3 = \Delta x \cdot f(2) + \Delta x \cdot f(3) + \Delta x \cdot f(4)$

$= 1 \cdot 3 + 1 \cdot 8 + 1 \cdot 15$

$= \boxed{26}$

−6 no answer

−4 attempt but not close

−1 for wrong Δx

−1 per wrong evaluation of $f(2), f(3), f(4)$.

(b) (2pts) Sketch the rectangles that you used in part (a) on the provided graph. all or none

(c) (2pts) Is your answer in (a) larger or smaller than the true area $\int_1^4 (x^2 - 1)\, dx$? Explain.

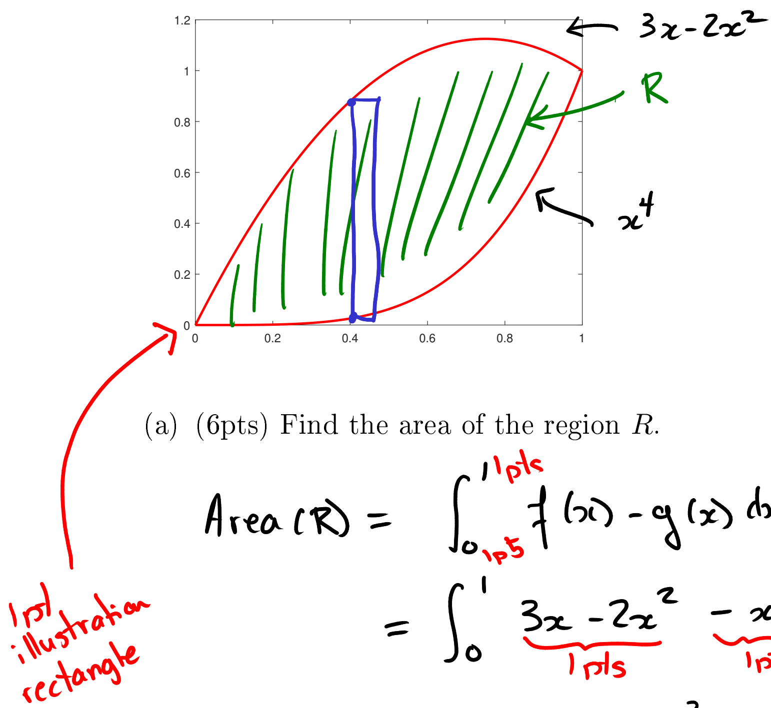
Larger

$f'(x) = 2x > 0$ for $x > 0$. So f is increasing on [1,2].
A right endpoint approximation of an increasing function yields an over-estimate.

−2 for "smaller"

−1 explanation doesn't mention $f(x)$ increasing, only justified by sketched rectangles

13. Consider the region R in the first quadrant bounded by the curves $y = x^4$ and $y = 3x - 2x^2$, pictured below. The two curves intersect at the points $(0, 0)$ and $(1, 1)$.



(a) (6pts) Find the area of the region R.

$$\text{Area}(R) = \int_{0}^{1} f(x) - g(x)\,dx \qquad (f(x) \geq g(x))$$

1pts 1pts

$$= \int_0^1 \underbrace{3x - 2x^2}_{1\text{pts}} \underbrace{- x^4}_{1\text{pts}}\,dx$$

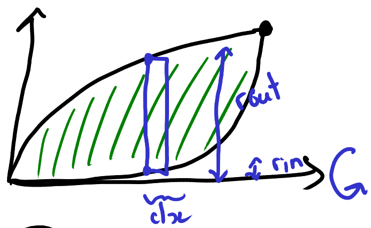
$$= \frac{3x^2}{2} - \frac{2x^3}{3} - \frac{x^5}{5}\Big|_0^1 \qquad 1\text{pt.}$$

$$= \boxed{\frac{19}{30}}$$

Problem continues on the next page...

(b) (4pts) Set up **but do not evaluate** an integral for the volume of the solid obtained by rotating R about the x-axis.

① Picture.



$r_{out} = 3x - 2x^2$ 1pt

$r_{in} = x^4$ 1pt

thikness $= dx$ 1pt

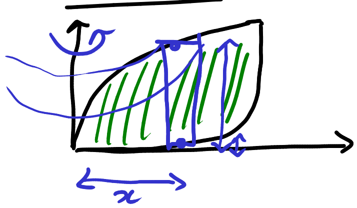
② Set-up.

$$\text{Vol} = \int_0^1 \pi \left(r_{out}^2 - r_{in}^2\right) dx$$

$$= \int_0^1 \pi \left(3x - 2x^2\right)^2 - \pi\, x^8 \, dx$$ 1pt

(c) (4pts) Set up **but do not evaluate** an integral for the volume of the solid obtained by rotating R about the y-axis.

① Picture.



radius $= x$ 1pt

hieght $= 3x - 2x^2 - x^4$ 1pt

thickness $= dx$ 1pt

② Set-up.

$$\text{Vol} = \int_0^1 2\pi \text{ radius} \cdot \text{hieght } dx$$

$$= \int_0^1 2\pi x \left(3x - 2x^2 - x^4\right) dx$$ 1pt.