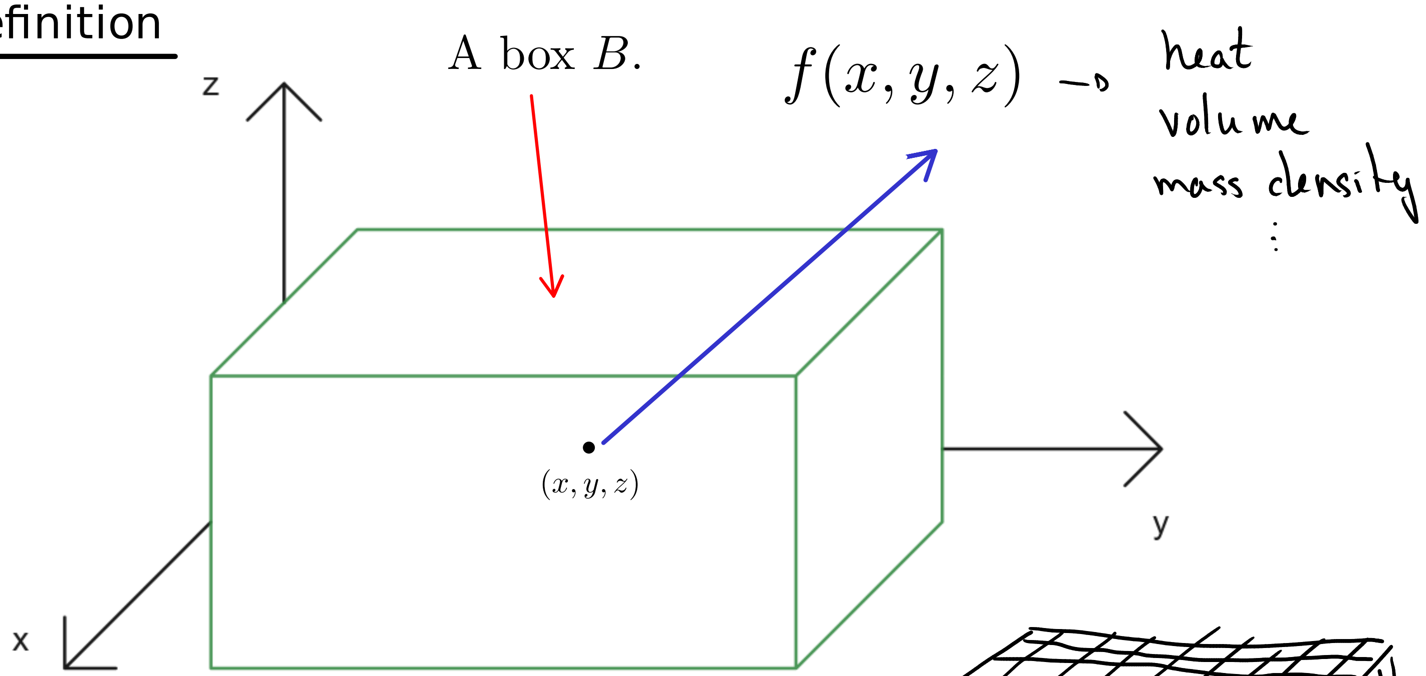
# Chapter 15

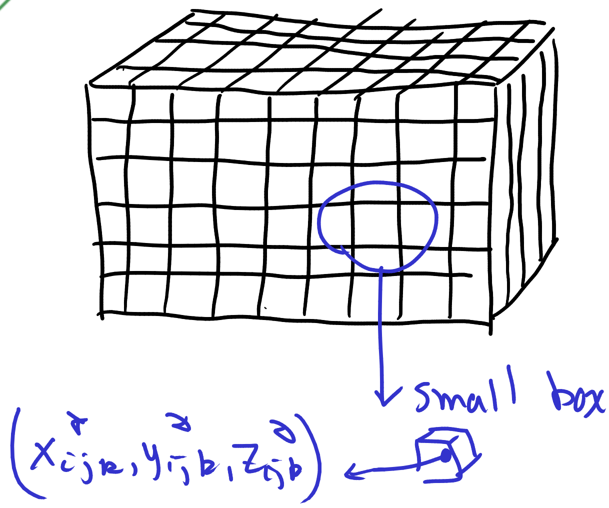
# Multiple Integrals

## 15.6 Triple integrals

## Definition




- A box $B = [a,b] \times [c,d] \times [r,s]$
- Divide [a, b] in $l$ parts
- Divide [c, d] in $m$ parts
- Divide [r, s] in $n$ parts




$$\mathrm{Vol}(\square) = \Delta V$$

Heat in a small box

$$\approx f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V$$

Total heat in box

$$\approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V$$

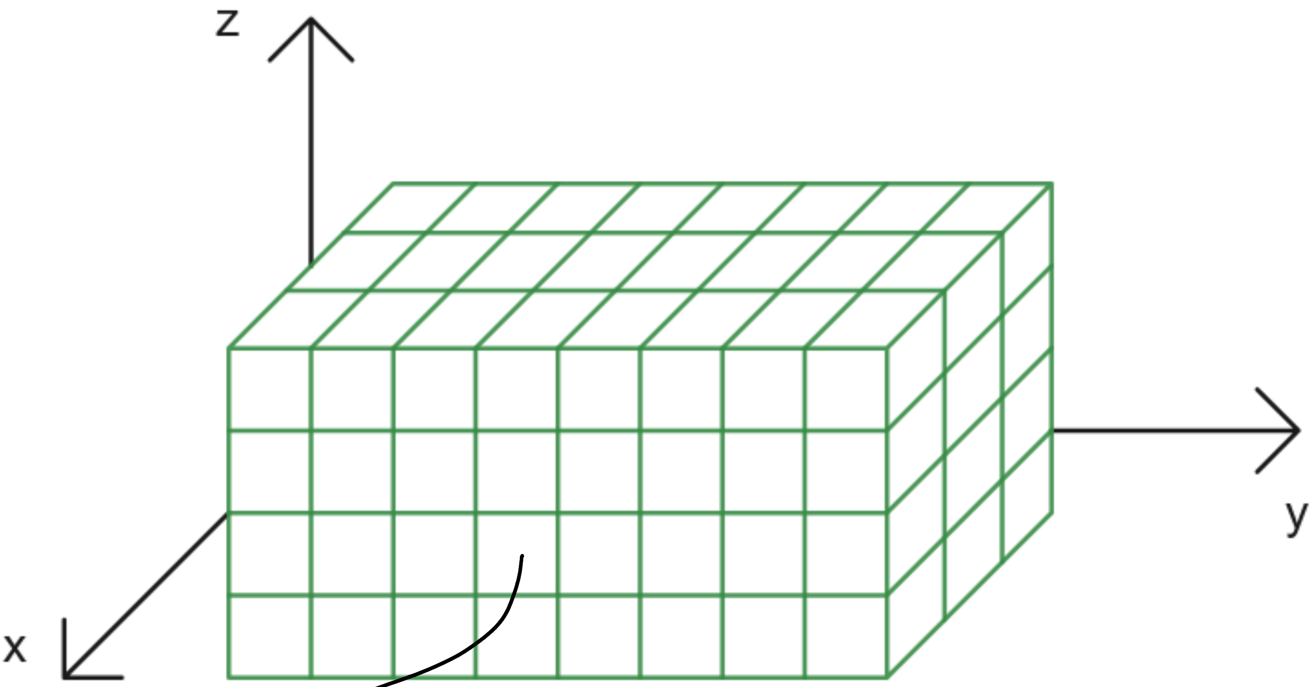
Take the limit as $l, m, n \longrightarrow \infty$

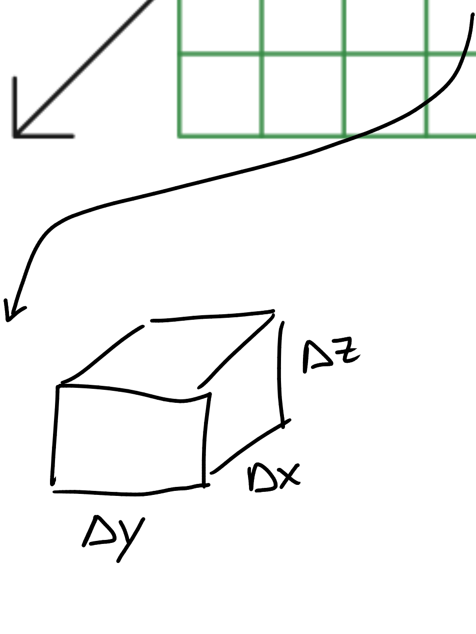
The triple integral of $f$ over the box $B$ is

$$\boxed{\iiint_B f(x,y,z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V}$$

# Triple integrals in cartesian coordinates

- Write explicitly $B = \{(x, y, z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$
- Divide $[a, b]$ in parts of length $\Delta x$.
- Divide $[c, d]$ in parts of length $\Delta y$.
- Divide $[r, s]$ in parts of length $\Delta z$.




$\Delta V = \Delta x \, \Delta y \, \Delta z$

As the # of parts $\to \infty$

$dV = dx \, dy \, dz$

## Fubini's Theorem for triple integrals

$$\boxed{\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx dy dz}$$

the function must be continuous on the box B.

**EXAMPLE 1** Evaluate the triple integral $\iiint_B xyz^2\,dV$, where $B$ is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

$$\iiint_B xyz^2\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz.$$

$$= \int_0^3 \int_{-1}^2 \frac{x^2}{2}\Big|_0^1 yz^2\,dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} yz^2\,dy\,dz$$

$$= \frac{1}{2} \int_0^3 \int_{-1}^2 yz^2\,dy\,dz$$

$$= \frac{1}{2} \int_0^3 \frac{y^2}{2}\Big|_{-1}^2 z^2\,dz$$

$$= \frac{3}{4} \int_0^3 z^2\,dz = \frac{3}{4}\,\frac{z^3}{3}\Big|_0^3 = \boxed{\frac{27}{4}}$$

Comment: $\left(\int_0^1 x\,dx\right)\left(\int_{-1}^2 y\,dy\right)\left(\int_0^3 z^2\,dz\right)$ is also another to calculate the integral.

QUESTION. What are the 5 other configurations of dx, dy, dz in a triple integral?

1. $dV = dz\,dy\,dx$
2. $dV = dz\,dx\,dy$
3. $dV = dy\,dx\,dz$
4. $dV = dy\,dz\,dx$
5. $dV = dx\,dz\,dy$

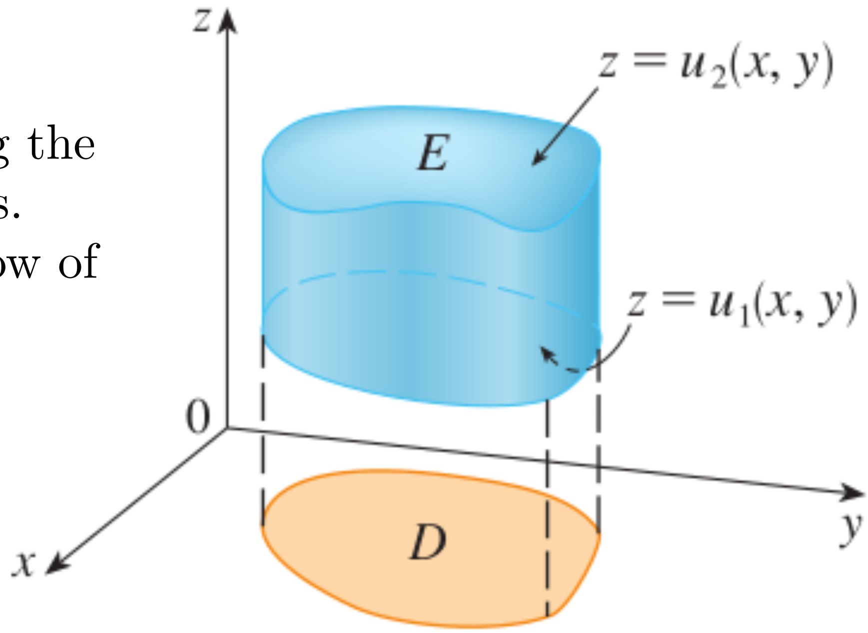
## General Domains.

For $E$ a general solid, let $B$ be a box containing $E$.
Define a function $F$ on $B$ :

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \in B \backslash E. \end{cases}$$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV}$$

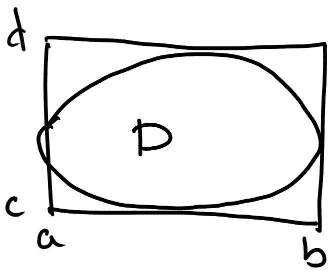
## Domain of type 1.




- Solid $E$ is bounded along the $z-$axis by two functions.
- Define $D$ to be the shadow of $E$ in the $xy-$plane.
- The domain $D$ can be of type I or type II.

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

$$= \int_a^b \int_c^d \int_r^s F(x,y,z) \quad dz\,dy\,dx$$

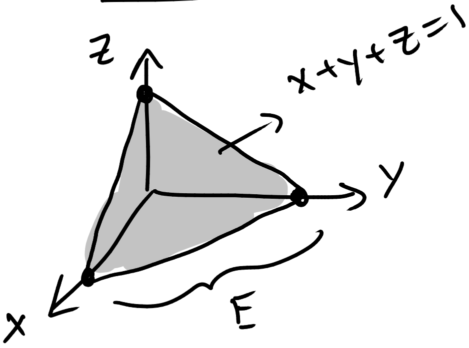



$$= \iint_D \left( \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[ \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA}$$

**EXAMPLE 2** Evaluate $\iiint_E \overset{f(x,y,z)}{z}\, dV$, where $E$ is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

① <u>Picture</u>




Describe E as a typ 1:

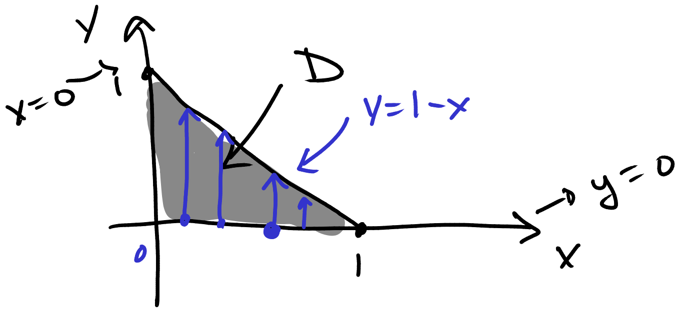
<u>z-values</u>

$z = 0$ (lower surface)

$z = 1 - x - y$

$$\Rightarrow \quad \underbrace{0}_{u_1(x,y)} \le z \le \overbrace{1-x-y}^{u_2(x,y)}.$$

<u>Shadow in xy-plane</u>




set $z=0 \Rightarrow x+y+0=1$

$\Rightarrow x+y=1$

D as a type I:

$$D = \{(x,y) : \; 0 \le x \le 1, \; 0 \le y \le 1-x\}$$

② <u>Integrate</u>

$$\iiint_E z\, dV = \iint_D \left( \int_0^{1-x-y} z\, dz \right) dA$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$

$$\longrightarrow -\frac{(1-x-y)^3}{6} \Big|_0^{1-x}$$

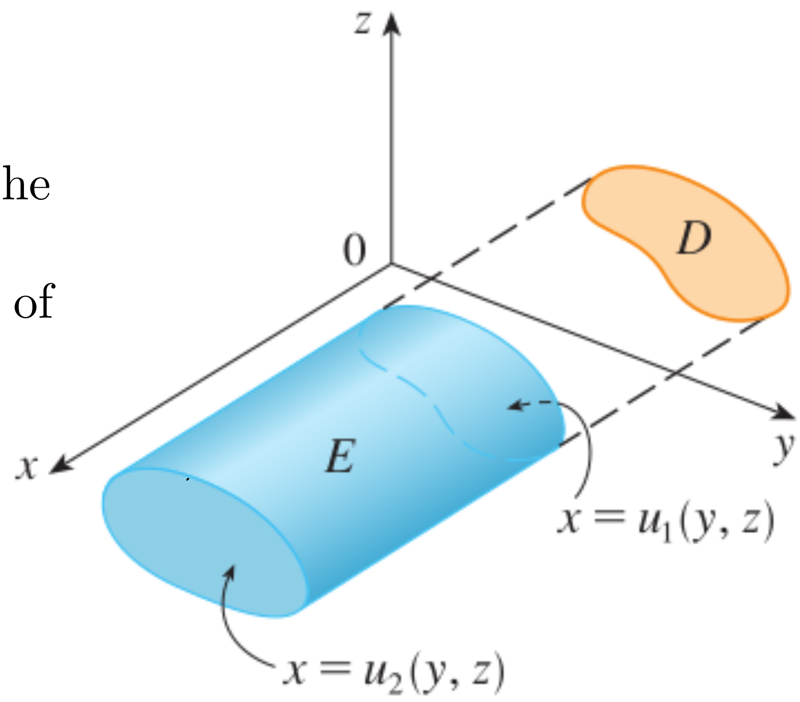
$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1-2x-2y+2xy+x^2+y^2}{2}\, dy\, dx$$

$$= \boxed{\frac{1}{24}} \approx 0.0417.$$

## Domains of type 2.

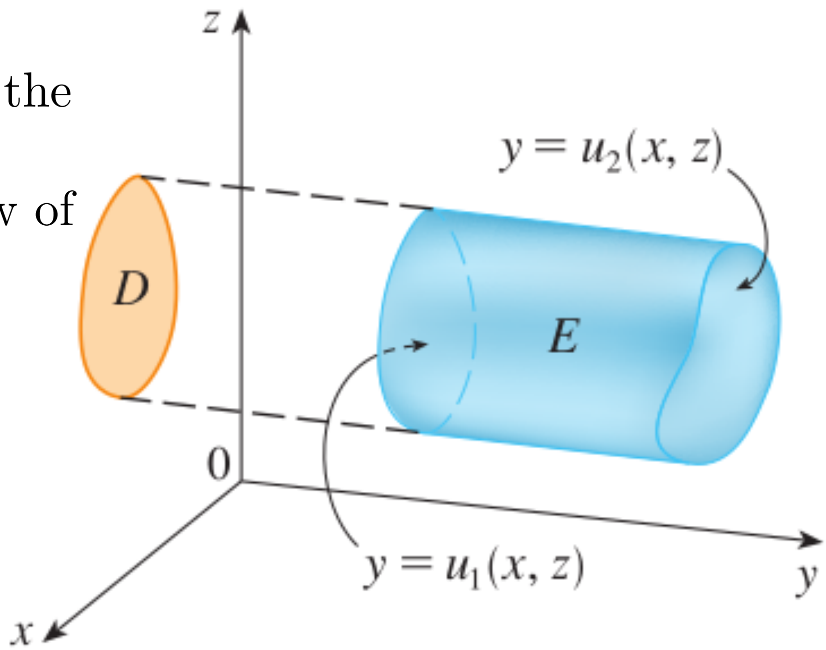
- Solid $E$ is bounded along the $x-$axis by two functions.
- Define $D$ to be the shadow of $E$ in the $yz-$plane.
- The domain $D$ can be of type I or type II.




$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

## Domains of type 3.

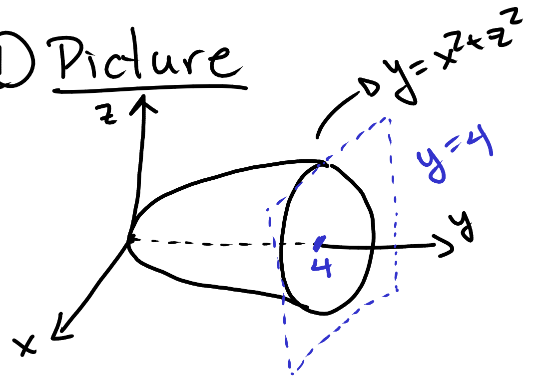
- Solid $E$ is bounded along the $y-$axis by two functions.
- Define $D$ to be the shadow of $E$ in the $xz-$plane.
- The domain $D$ can be of type I or type II.




$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

**EXAMPLE 3** Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where $E$ is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

① Picture




Type 3: y-values are bounded by two functions.

y-values: $x^2+z^2 \le y \le 4$.

Shadow in xz-plane

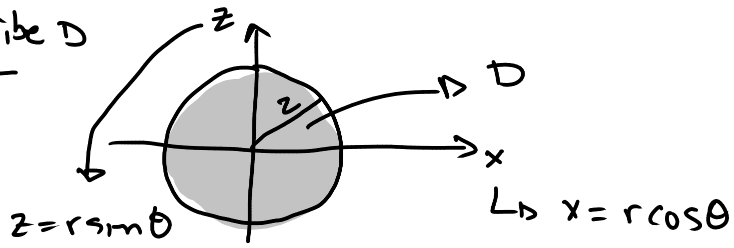
Set $y=4 \Rightarrow 4 = x^2+z^2$

↳ circle radius $= 2$.

$x = r\cos\theta$, $z = r\sin\theta$

$D = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

Describe D ←




② Integrate

$$\iiint_E (x^2+z^2)^{1/2}\,dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right) dA$$

$$= \iint_D \sqrt{x^2+z^2}\,\left(4-(x^2+z^2)\right) dA$$

$$= \int_0^{2\pi}\int_0^{2} r\,(4-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 4r^2 - r^4\,dr\,d\theta$$

$$= \left(\int_0^2 4r^2 - r^4\,dr\right)\left(\int_0^{2\pi} 1\,d\theta\right)$$

$$= \boxed{\frac{128\pi}{15}} \approx 26.808.$$

**EXAMPLE 4** Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to $x$, then $z$, and then $y$.

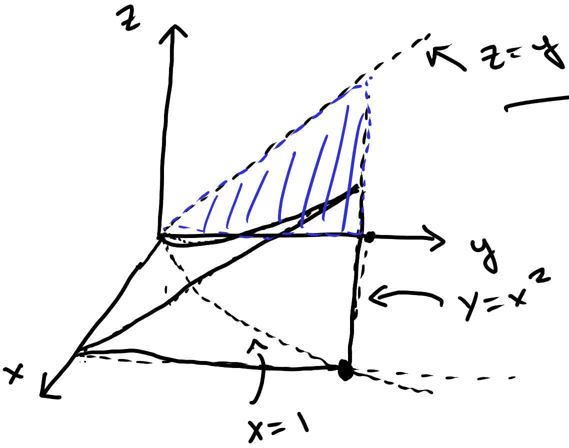
$= I$

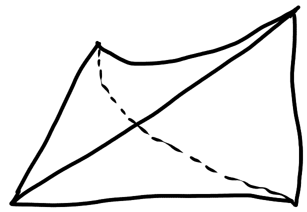
Goal: $dx\,dz\,dy$.

① Picture

→ Type 1

$$E = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$$



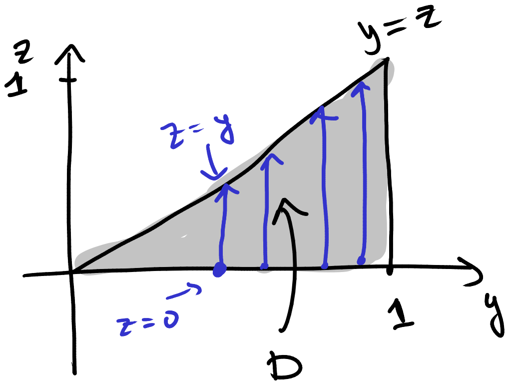



② Change order    Switch to a type 2.

x-values    coming in : $x = \sqrt{y}$  $(y = x^2)$
coming out : $x = 1$

$\sqrt{y} \le x \le 1$

yz-plane.




D as a type I

$$D = \{(y,z) : 0 \le y \le 1,\ 0 \le z \le y\}$$

$$E = \{(x,y,z) : \sqrt{y} \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le y\}$$

$$I = \iiint_E f(x,y,z)\, dV$$

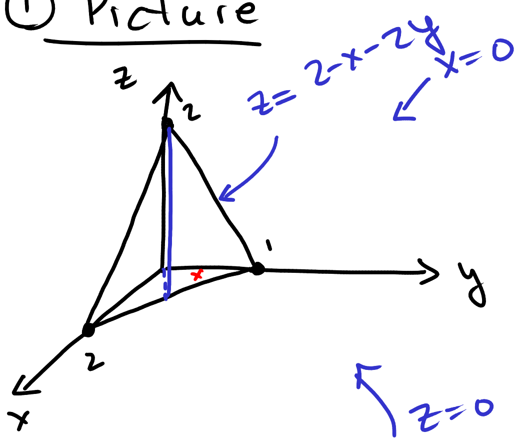
$$= \boxed{\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy}$$

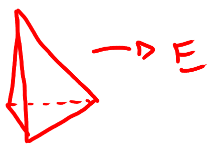
## Application: computing volumes of solids.

$$\boxed{\mathrm{Vol}(E) = \iiint_E 1\,dV}$$

**EXAMPLE.** Use a triple integral to find the volume of the tetrahedron $T$ bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

① Picture



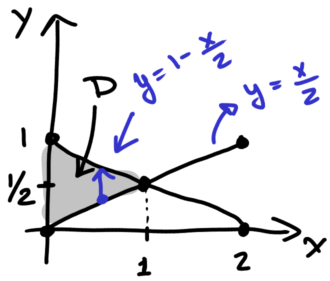

Type 1

z-values $\qquad 0 \le z \le 2 - x - 2y$

xy-plane

$z = 0 \to x + 2y = 2$

$x = 2y \to y = \frac{x}{2}$




$$D = \left\{(x,y) :\ 0 \le x \le 1,\ \frac{x}{2} \le y \le 1 - \frac{x}{2}\right\}$$

② Volume

$$\mathrm{Vol}(E) = \iiint_E 1\,dV$$

$$= \iint_D \left( \int_0^{2-x-2y} dz \right) dA$$

$$= \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} dz\,dy\,dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} 2 - x - 2y\,dy\,dx = \boxed{\frac{1}{3}}$$