# Chapter 15

# Multiple Integrals

## 15.9 Change of variables in multiple integrals

## Change of variable from Calculus I

$u = g(x)$

$du = g'(x)\,dx$

If $x = g(u)$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$.

## Change of Variable in polar coordinate.

If $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\iint_D f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta) r\,dr d\theta$$

where $R$ is a region in the $xy$-plane and $S$ is a region in the $r\theta$-plane.

Transformation in polar coordinates:

$$(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta).$$

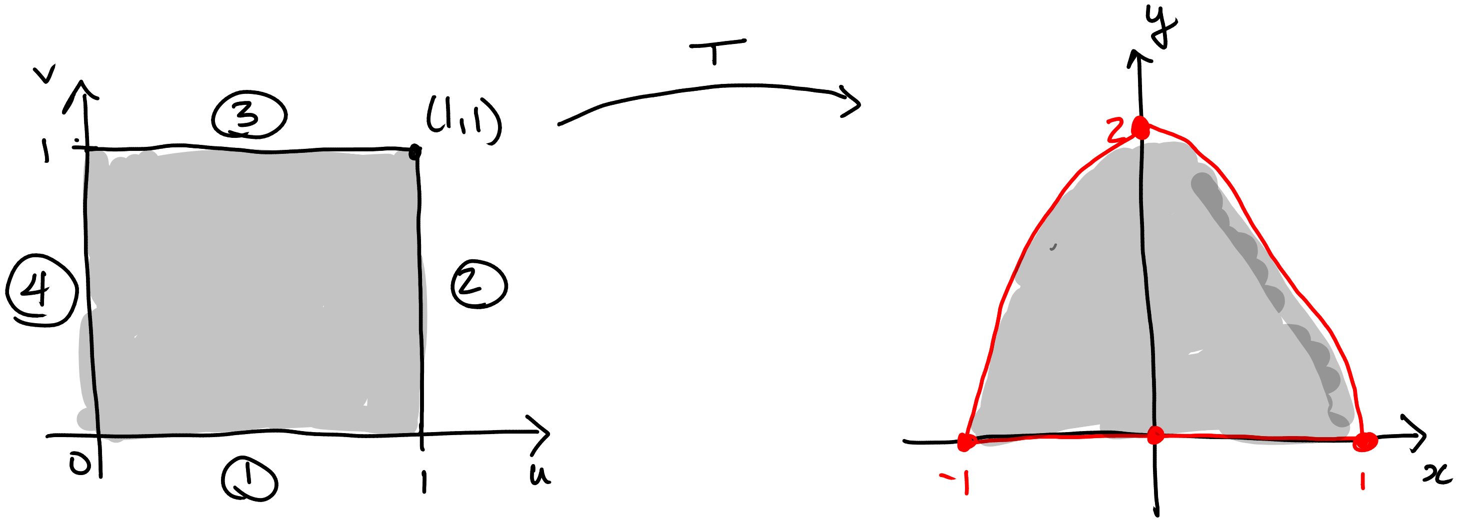
## General transformation in 2D.

**EXAMPLE 1** A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u,v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

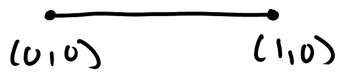
$$T(u,v) = (x,y) = (u^2 - v^2, 2uv).$$



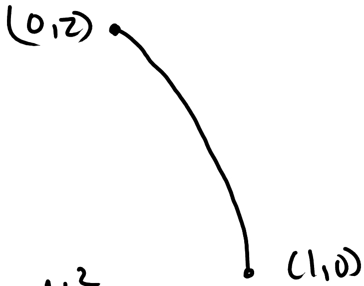
① $\underline{0 \le u \le 1, \ v = 0}$

$$x = u^2 - v^2 = u^2 - 0^2 = u^2$$

$$y = 2uv = 2 \cdot u \cdot 0 = 0$$




② $\underline{u = 1, \quad 0 \le v \le 1}$




$y^2 = 4 - 4x^2$

$x = 1 - v^2$

$y = 2v \quad \rightarrow \quad v = \frac{y}{2} \quad \rightarrow \quad x = 1 - \frac{y^2}{4}, \ 0 \le y \le 2$

③ $\underline{0 \le u \le 1, \quad v = 1}$

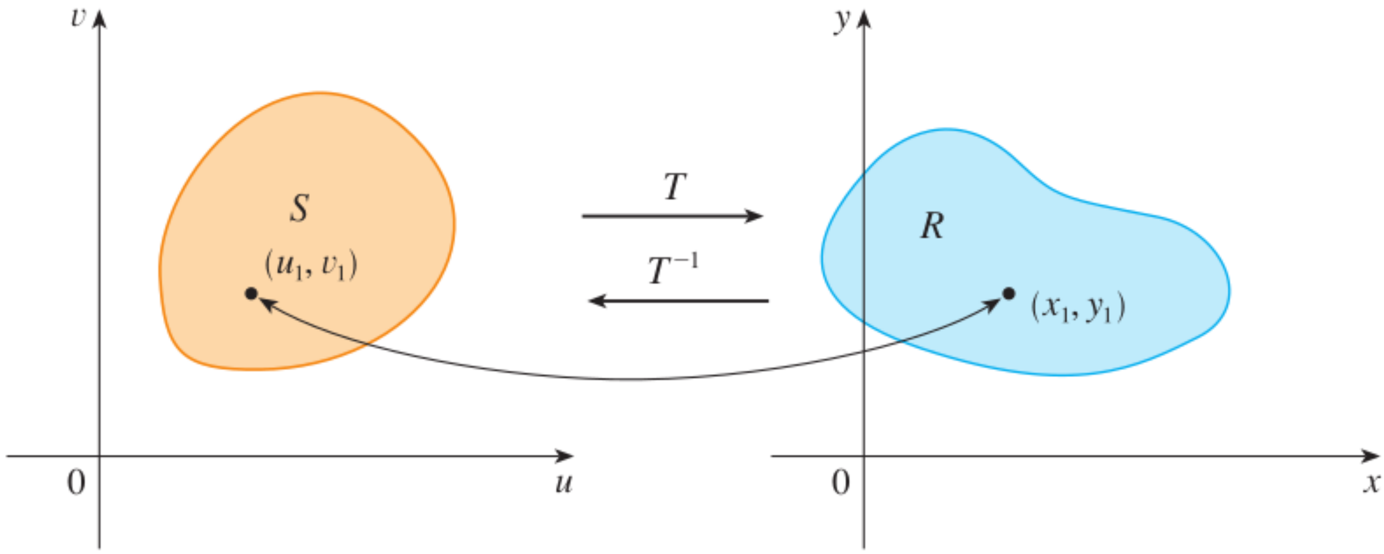
$x = u^2 - 1 \quad \rightarrow \quad u = \frac{y}{2} \quad \rightarrow \quad x = \frac{y^2}{4} - 1, \quad 0 \le y \le 2$

$y = 2u$

④ $\underline{u = 0, \ 0 \le v \le 1}$

$x = -v^2 \quad \rightarrow \quad -1 \le x \le 0$

$y = 0$



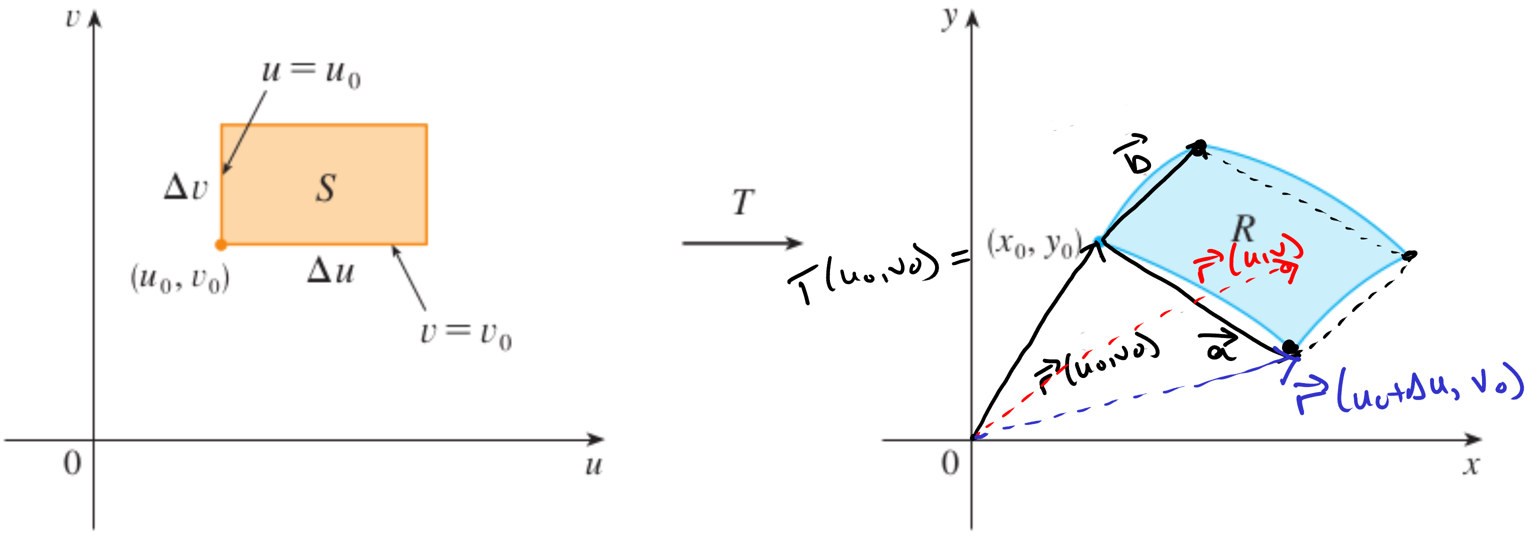

Two equations for x and y:

$$(x, y) = T(u, v) \quad \Longleftrightarrow \quad x = x(u, v) \text{ and } y = y(u, v)$$

Image: The region $R$ is the set of possible outputs.

Domain: The region $S$ is the set of all possible inputs.

Effect of a change of variables in double integral.




Goal: Find how dA is transformed after the transformation.

$$\text{Area}(R) \approx \|\vec{a} \times \vec{b}\| .$$

Recall
$$\vec{r}_u(u_0,v_0) = \lim_{\Delta u \to 0} \frac{\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)}{\Delta u} .$$
$$\vec{r}_v(u_0,v_0) = \lim_{\Delta v \to 0} \frac{\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)}{\Delta v} .$$

When $\Delta u$ and $\Delta v$ are small :

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0) \approx \Delta u \, \vec{r}_u(u_0,v_0) .$$
$$\vec{b} \approx \Delta v \, \vec{r}_v(u_0,v_0)$$

So,
$$\text{Area}(R) \approx \|\vec{r}_u \times \vec{r}_v\| \, \Delta u \, \Delta v$$

$$\Rightarrow \quad \Delta A \approx \|\vec{r}_u \times \vec{r}_v\| \, \Delta u \, \Delta v \qquad \left( \Delta y \approx f'(a) \, \Delta x \right)$$

Make $\Delta u, \Delta v \to 0$ (or nb. of divisions $\to \infty$):

$$\Rightarrow \quad dA = \underbrace{\|\vec{r}_u \times \vec{r}_v\|}_{\substack{\text{more explicit} \\ \text{expression} \\ \text{please.}}} \, du \, dv$$

Here, $\vec{r}(u,v) = T(u,v) = (x(u,v), y(u,v)) \hookrightarrow$ 3D

$\Rightarrow \vec{r}_u = \langle x_u , y_u , 0\rangle , \quad \vec{r}_v = \langle x_v , y_v , 0\rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \underbrace{(x_u y_v - x_v y_u)}_{\text{Jacobian } \frac{\partial(x,y)}{\partial(u,v)}} \vec{k}$$

$$\Rightarrow \quad dA = |x_u y_v - x_v y_u| \, du\, dv .$$

Useful notation: $\quad \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du}$$

type I

or

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

type II

Remarks:

① $\dfrac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ is called the <u>Jacobian</u>. of the transform. $T$.

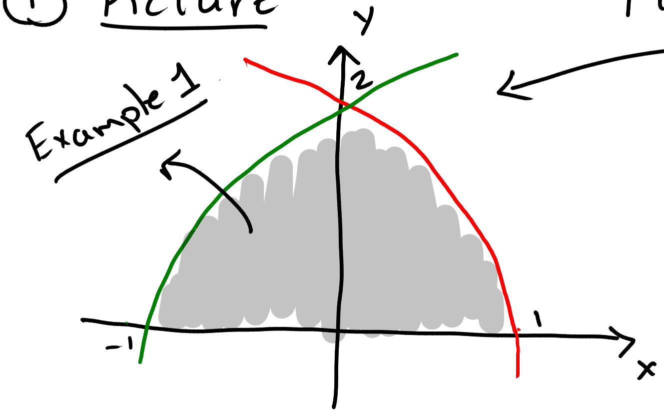
② If $T^{-1}$ exists, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)} .$$

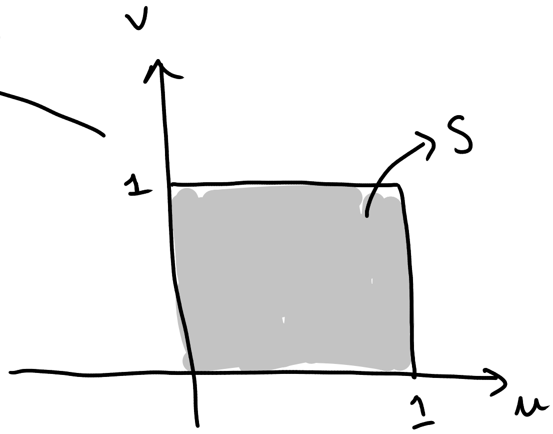
③ The formulas for $\dfrac{\partial(x,y)}{\partial(u,v)}$ and $dA$ work when $T$ is a $C^1$-transformation (this means the derivatives exist and are continuous).

**EXAMPLE 2** Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where $R$ is the region bounded by the $x$-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

① Picture




$T(u,v) = (x,y)$




$S = [0,1] \times [0,1]$ and $T(S) = R$

$\boxed{\begin{aligned} x &= u^2 - v^2 \\ y &= 2uv \end{aligned}}$

② Integrate

$$\iint_R y\, dA = \iint_S 2uv \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{??} du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

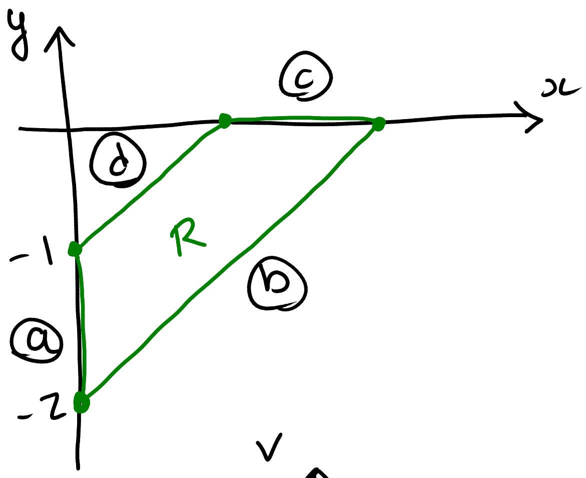
$$= \int_0^1 \int_0^1 2uv\,(4u^2 + 4v^2)\, du\,dv$$

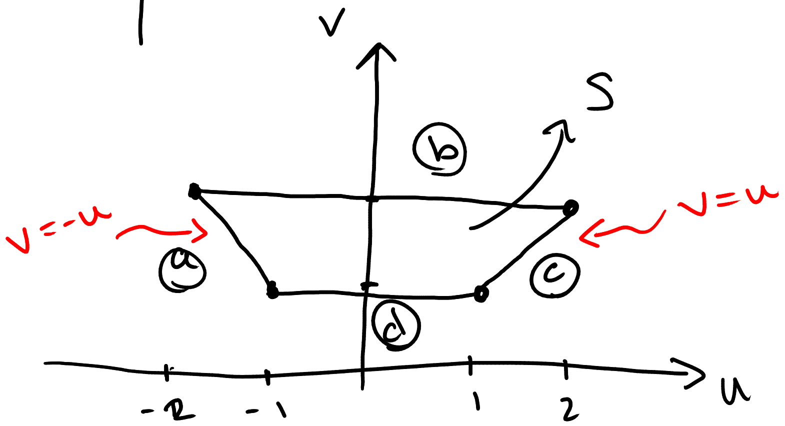
$$= \int_0^1 \int_0^1 8u^3 v + 8uv^3 \, du\,dv$$

$$= \boxed{2}$$

**EXAMPLE 3** Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where $R$ is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

① Picture




② Change of var.

$u = x+y, \quad v = x-y$

ⓐ $x=0, \quad -1 \le y \le -2$

$u = y, \quad v = -y$

$\Rightarrow v = -u$

$-2 \le u \le -1$

ⓑ $0 \le x \le 2, \quad y = x-2$

$u = 2x-2, \quad v = 2$

ⓒ $u = v, \quad 1 \le u \le 2$

ⓓ $-1 \le u \le 1, \quad v = 1$

② Integrate

$$\iint_R e^{(x+y)/(x-y)}\,dA = \iint_S e^{u/v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$S = \{(u,v) : -v \le u \le v, \quad 1 \le v \le 2\}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

So,

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{|-2|} = \frac{1}{2}$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2}\right) du\, dv$$

$$\boxed{\int e^{x/2} dx = 2e^{x/2}}$$

$$= \frac{1}{2} \int_1^2 v e^{u/v} \Big|_{-v}^{v} dv$$

$$= \frac{1}{2} \int_1^2 v \left( e^1 - e^{-1} \right) dv$$

$$= \frac{e - e^{-1}}{2} \int_1^2 v \, dv$$

$$= \frac{e - e^{-1}}{2} \quad \frac{v^2}{2} \Big|_1^2$$

$$= \boxed{\frac{3}{4} \left(e - e^{-1}\right)} \approx 1.7628.$$

## Effect of change of variable in Triple integrals.

Spherical coordinates.

$$(x, y, z) = T(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$

This implies that

$$dV = \underbrace{\rho^2 \sin\phi}_{\text{Jacobien of the transformation.}}\, d\rho\, d\theta\, d\phi$$

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Transformation in 3D:

- A function $T$ from a region $S$ in the $uvw$-space into a region $R$ in the $xyz$-space.
- So

$$(x, y, z) = T(u, v, w)$$
$$\Updownarrow$$
$$x = x(u, v, w),\ y = y(u, v, w) \text{ and } z = z(u, v, w)$$

Jacobian in 3D:

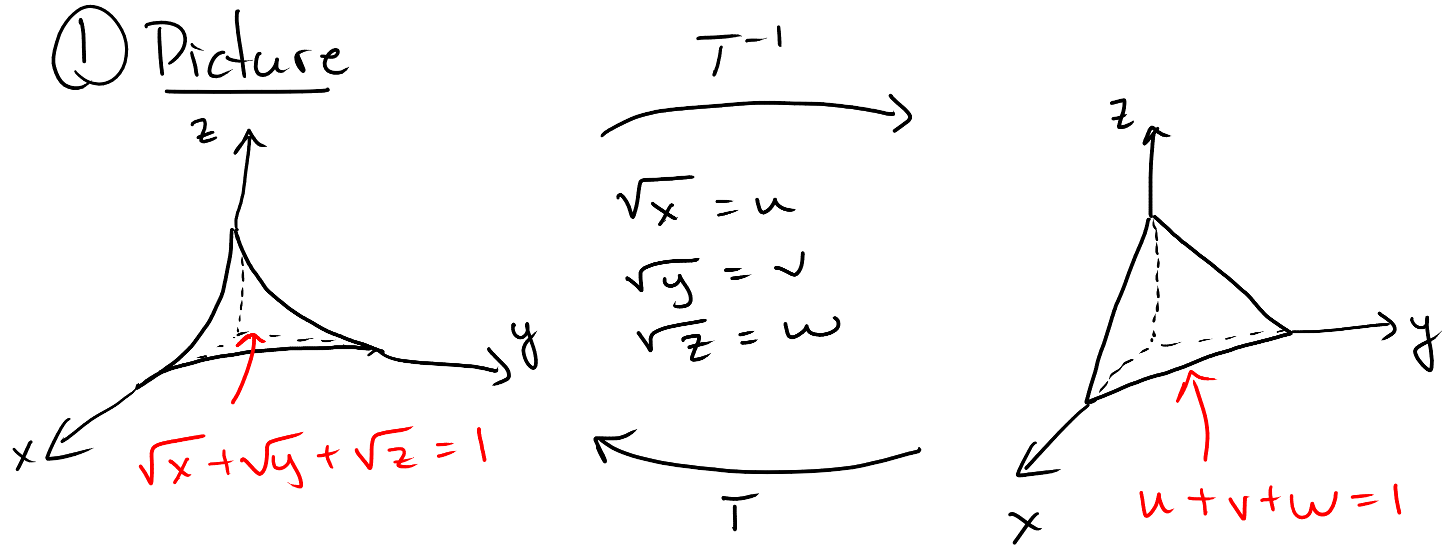
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\boxed{\iiint\limits_R f(x, y, z)\, dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw}$$

Important fact: If $T^{-1} : R \to S$ exists, then $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$

**56.** Use the transformation $x = u^2,\ y = v^2,\ z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

① Picture




$S$ as type 1:

$$S = \{(u,v,w) : 0 \le u \le 1,\ 0 \le v \le 1-u,\ 0 \le w \le 1-u-v\}$$

② Integrate

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} dw\,dv\,du$$

$$= 8uvw\,dw\,dv\,du$$

So,

$$\mathrm{Vol}(R) = \iiint_R 1\,dV = \iiint_S 8uvw\,dw\,dv\,du$$

$$\Rightarrow \quad \mathrm{Vol}(R) = \int_0^1 \int_0^{1-u} \int_0^{1-u-w} 8uvw \; dw\,dv\,du$$

$$= \boxed{\frac{1}{90}} \approx 0.011 \, .$$