Math 244

Chapter 16

Section 16.8: Stokes' Theorem

Contents

Created by: Pierre-Olivier Parisé

Fall 2023

DEFINITION 1. If $\vec{F} = \langle P, Q, R \rangle$ is a vector field in 3D, then

$$\text{curl}\,\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle\,.$$

Another way to write curl $\vec{F}$ is as followed. Define

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \Longrightarrow \quad \text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F}.$$

EXAMPLE 1. Find the curl of $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

SOLUTION.

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz), -\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right), \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \right\rangle$$

$$= \langle -2y - xy, x, yz \rangle$$

Theorem 1. Let $\vec{F} = \langle P, Q, R \rangle$. If

- P, Q, R have continuous partial derivatives.
- curl $\vec{F} = \vec{0}$.

Then $\vec{F}$ is conservative.

Example 2. Let $\vec{F}(x, y, z) = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$.

a) Show that $\vec{F}$ is conservative.

b) Find a function f such that $\vec{F} = \vec{\nabla} f$.

Solution.

a) $$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix}$$

$$= \langle 6xyz^2 - 6xyz^2, -3y^2z^2 + 3y^2z^2, 2yz^3 - 2yz^3 \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}.$$

$\Rightarrow$ $\vec{F}$ is conservative.

b) $\vec{\nabla} f = \vec{F}$ $\Rightarrow$ $$\begin{cases} f_x = y^2 z^3 \\ f_y = 2xyz^3 \\ f_z = 3xy^2 z^2 \end{cases}$$

Integrate all the equations:

$$\begin{cases} f(x,y,z) = xy^2z^3 \\ f(x,y,z) = xy^2z^3 \\ f(x,y,z) = xy^2z^3 \end{cases}$$

Final expression:

$$\boxed{f(x,y,z) = xy^2z^3 + c}$$

Stokes' Theorem

Recall Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA,$$

where C is orientated positively. Writing $\vec{F} = \langle P, Q, 0 \rangle$:

$$Q_x - P_y = \langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = \operatorname{curl} \vec{F} \cdot \vec{k}$$

so that

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \vec{F} \cdot \vec{k} \, dA.$$

A particular case of Stokes' Theorem.

Theorem 2. Assume

- S be an oriented surface bounded by a loop C with orientation induced by the surface.
- $\vec{F} = \langle P, Q, R \rangle$ with P, Q, R having continuous partial derivatives.

Then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

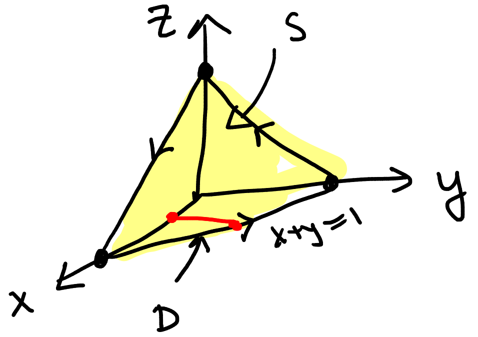
Rule of Thumb: What we mean by the orientation induced by the surface is: we apply the right-hand rule with the thumb pointing in the direction of the normal vector.

EXAMPLE 3. Let $\vec{F}(x,y,z) = \langle x + y^2, y + z^2, z + x^2 \rangle$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Compute

$$\int_C \vec{F} \cdot d\vec{r}.$$

SOLUTION.

① <u>Picture</u>



Plane as S : $x + y + z = 1$

$$\vec{r}(u,v) = \langle u, v, 1-u-v \rangle$$

$$D = \{ (u,v) : 0 \leq u \leq 1, \ 0 \leq v \leq 1 - x \}$$

② <u>Stokes' Theorem</u>

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} = \langle -2z, -2x, -2y \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

Then,

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \langle -2(1-u-v), -2u, -2v \rangle \cdot \langle 1,1,1 \rangle \, dA$$

$$= \int_0^1 \int_0^{1-u} -2 + \cancel{2}u + \cancel{2}v - \cancel{2}u - \cancel{2}v \; dv\, du$$

$$= \int_0^1 \int_0^{1-u} -2 \; dv\, du$$

$$= -2 \int_0^1 1-u \, du$$

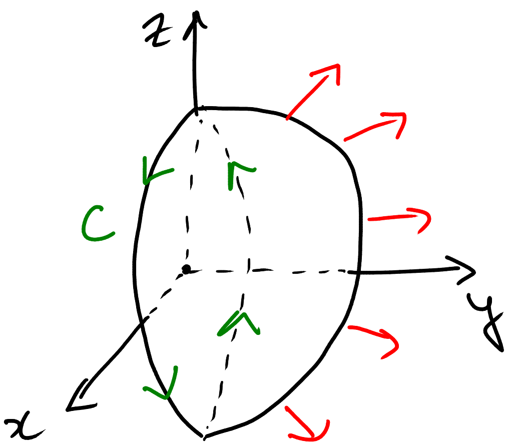
$$= -2\left(\frac{1}{2}\right) = \boxed{-1}$$

EXAMPLE 4. Let $\vec{F}(x,y,z) = \langle ze^y, x\cos y, xz\sin y\rangle$ and S be the hemisphere $x^2+y^2+z^2=16$, $y \geq 0$ oriented in the direction of the positive y-axis. Compute

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

SOLUTION.

Picture



C: circle $x^2+z^2=16$ with counterclockwise orientation.

$\vec{r}(t) = \langle 4\cos t, 0, 4\sin t\rangle$ with $0 \leq t \leq 2\pi$.

Stokes' Theorem

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

Then

$d\vec{r} = \vec{r}\,'(t)\,dt$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 4\sin t, 4\cos t, 0\rangle \cdot \langle -4\sin t, 0, 4\cos t\rangle\, dt$$

$$= \int_0^{2\pi} -16 \sin^2 t \, dt$$

$$= -16 \int_0^{2\pi} \frac{1 - \cos 2t}{2} \, dt$$

$$= \boxed{-16\pi}$$