$u = x^2 \to \boxed{du = 2x\,dx}$

$\int_0^1 x\cos(x^2)\,dx = \int_0^1 \cos(u)\,\frac{du}{2}$

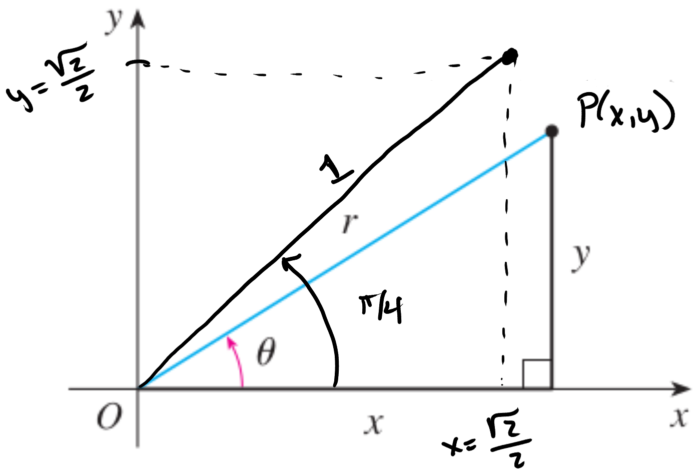
Chapter 15

Multiple Integrals

15.3 Double Integrals in polar coordinates

Polar coordinates

$r = 1$, $\theta = \frac{\pi}{4}$ $\longrightarrow$ $x = 1\cos(\pi/4) = \sqrt{2}/2$, $y = 1\sin(\pi/4) = \sqrt{2}/2$



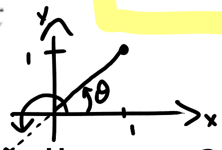
1) Polar to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin\theta$$

2) Cartesian to Polar:

$$r = \sqrt{x^2 + y^2}$$

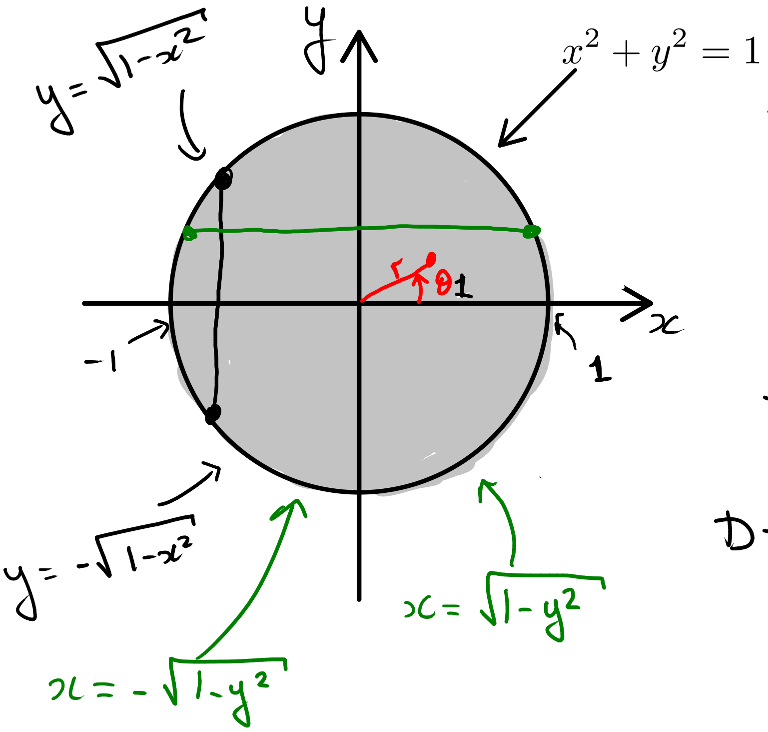
$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right) \quad \left(\theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$$



$x = 1$, $y = 1$ $\longrightarrow$ $r = \sqrt{2}$, $\theta = \boxed{\pi/4}, \cancel{5\pi/4}, \ldots$

Why would we use polar coordinates?

Example. Describe the following region:



TYPE I:

$$D = \{(x,y) : -1 \le x \le 1 \text{ and } -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$$

TYPE II:

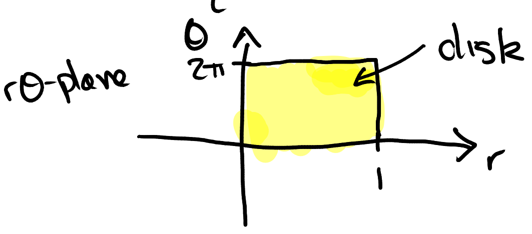
$$D = \{(x,y) : -1 \le y \le 1 \text{ and } -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}\}$$

Polar coordinates: $0 \le$ Distance from origin ≤ 1

$\qquad\qquad\qquad\qquad\qquad\qquad$ $\parallel$ r

$0 \le$ angle $= \theta \le 2\pi$

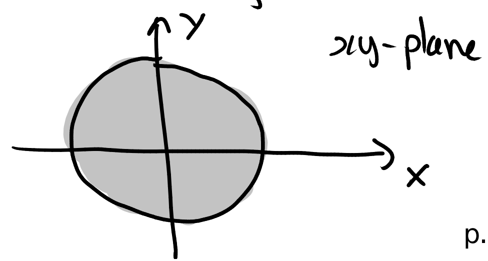
$$D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\} \longrightarrow \text{rectangle.}$$



$\xrightarrow{\hspace{2cm}}$

$x = r\cos\theta$

$y = r\sin\theta$



How does it affect the double integral

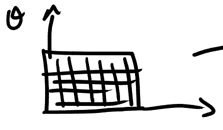
$u = x^2$
$du = \boxed{2x}\,dx$

$dA = ??$
$\boxed{}\,dr\,d\theta$

Recall:

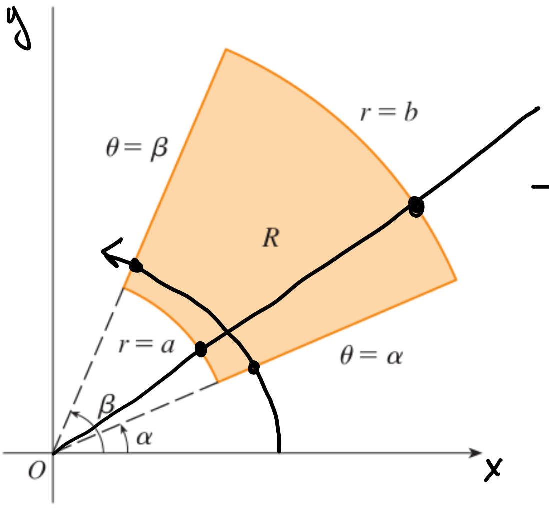
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy \longrightarrow \boxed{dA = dx\,dy}$$

$$= \int_c^d \int_a^b f(x,y)\,dy\,dx \longrightarrow \boxed{dA = dy\,dx}$$

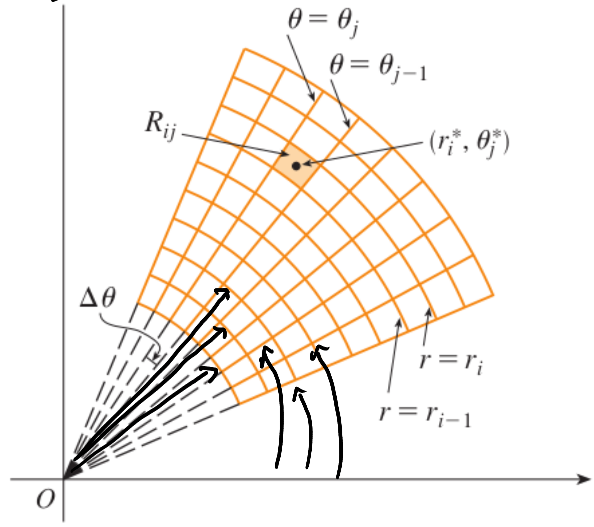


$x = r\cos\theta$
$y = r\sin\theta$

Polar rectangle:

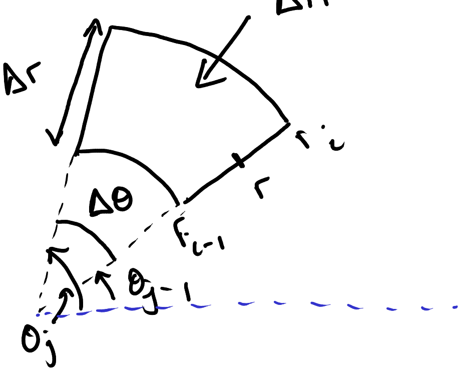


Divide it



Close-up view

$\Delta A = \text{area}(R_{ij})$



$$\Delta A = \frac{\Delta\theta \cdot r_i^2}{2} - \frac{\Delta\theta \cdot r_{i-1}^2}{2}$$

$$= \frac{\Delta\theta}{2}\left(r_i^2 - r_{i-1}^2\right)$$

$$= \frac{\Delta\theta}{2}(r_i - r_{i-1})(r_i + r_{i-1})$$

$$= \Delta\theta\,\Delta r\left(\frac{r_i + r_{i-1}}{2}\right)$$

$$= \Delta\theta\,\Delta r \cdot r$$

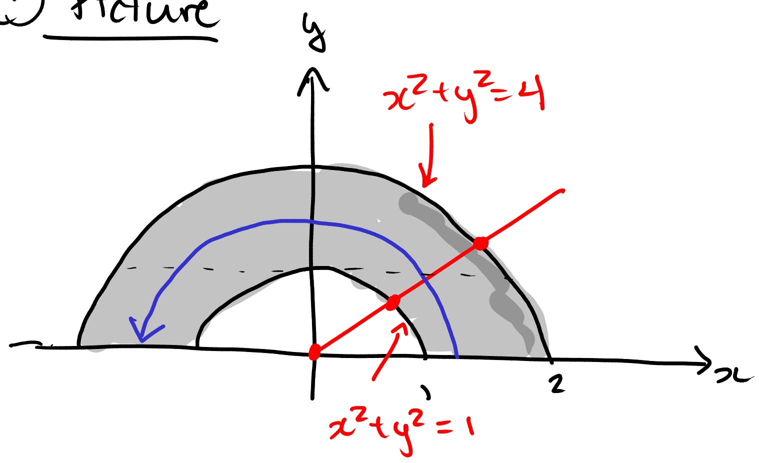
$$\Rightarrow \boxed{\Delta A = r\,\Delta\theta\,\Delta r} \longrightarrow \boxed{\begin{array}{l} dA = r\,d\theta\,dr \\ dA = r\,dr\,d\theta \end{array}}$$

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

R is a polar rectangle given by $a \le r \le b$ and $\alpha \le \theta \le \beta$, with $\beta - \alpha \le 2\pi$.

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

① Picture



Polar coord.

$$D = \{ (r, \theta) : \begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le \pi \end{array} \}$$

$$= [1, 2] \times [0, \pi]$$

$$x^2 + y^2 = 4 \to r^2 = 4 \to r = 2$$

$$x^2 + y^2 = 1 \to r^2 = 1 \to r = 1$$

② Integrate

$$\iint_R 3x + 4y^2\, dA = \int_0^\pi \int_1^2 (3 r\cos\theta + 4 r^2 \sin^2\theta)\, r\, dr\, d\theta$$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$$= \int_0^\pi \int_1^2 3r^2 \cos\theta + 4r^3 \sin^2\theta \, dr\, d\theta$$

$$= \int_0^\pi r^3 \cos\theta + r^4 \sin^2\theta \Big|_1^2$$

$$= \int_0^\pi 8\cos\theta + 16\sin^2\theta - \cos\theta - \sin^2\theta \, d\theta$$

$$= \int_0^\pi 7\cos\theta + 15\sin^2\theta \, d\theta$$

$$= 7\sin\theta \Big|_0^\pi + 15 \int_0^\pi \frac{1 - \cos 2\theta}{2}\, d\theta$$

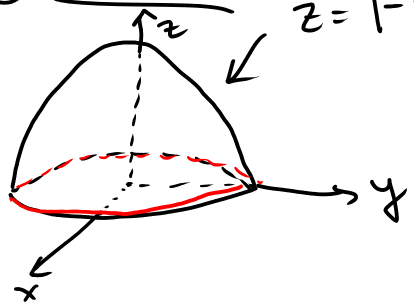
$$= \cancel{7 \cdot 0}^{\,0} + 15 \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) \Big|_0^\pi$$

$$= \boxed{\frac{15\pi}{2}}$$

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

① Picture

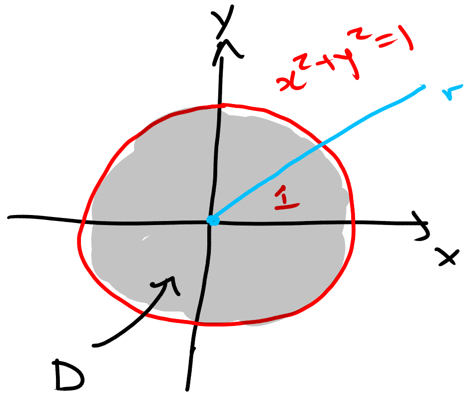
$z = 1 - x^2 - y^2$



$\xrightarrow[\text{on } z=0]{\text{project}}$

$0 = 1 - x^2 - y^2$

$\Rightarrow x^2 + y^2 = 1$



$D = \{(r,\theta) : 0 \le r \le 1 \,,\; 0 \le \theta \le 2\pi\}.$

② Volume $\quad f(x,y) = 1 - x^2 - y^2$

$$\mathrm{Vol}(S) = \iint_D 1 - x^2 - y^2 \, dA$$

$$= \int_0^{2\pi} \int_0^1 \left(1 - (r\cos\theta)^2 - (r\sin\theta)^2\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(1 - r^2\cos^2\theta - r^2\sin^2\theta\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(1 - r^2(\cos^2\theta + \sin^2\theta)\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta$$

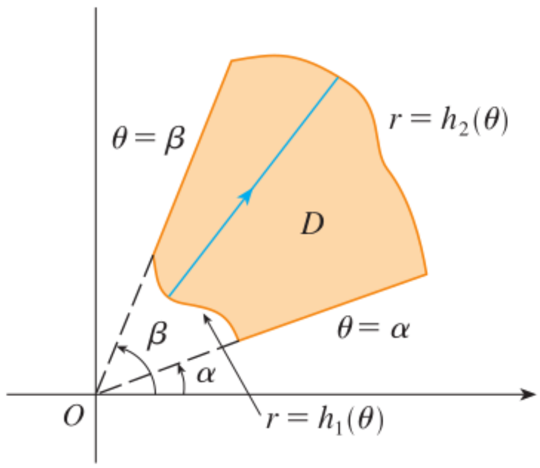
$$\int_a^b \int_c^d f(x)g(y) \, dx \, dy = \left(\int_c^d f(x)\, dx\right)\left(\int_a^b g(y)\, dy\right)$$

$$= \left(\int_0^{2\pi} 1 \, d\theta\right)\left(\int_0^1 r - r^3 \, dr\right).$$

$$= 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1$$

$$= \boxed{\frac{\pi}{2}}$$

More complicated region:



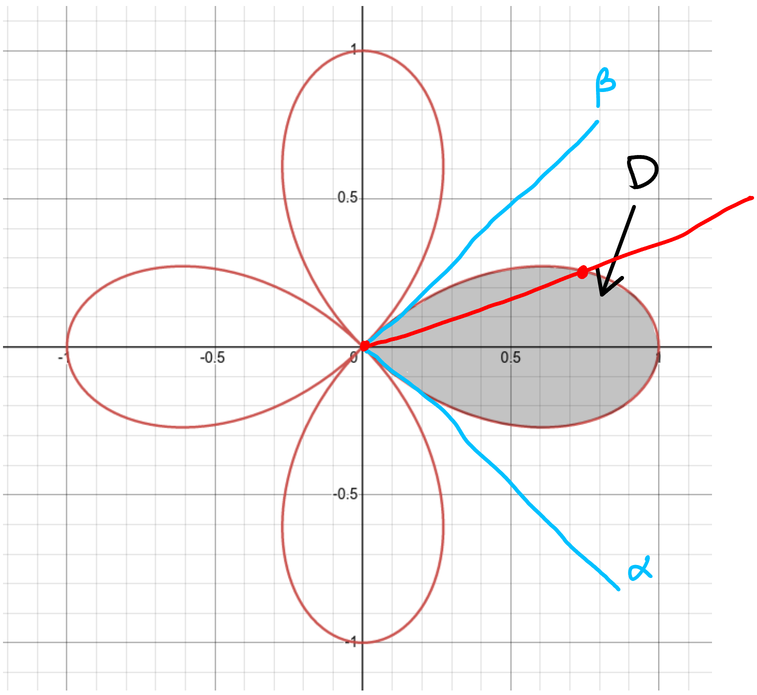
3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

1 PICTURE



$\alpha = -\pi/4$ $\qquad \beta = \pi/4$

$h_1(\theta) = 0$ $\qquad h_2(\theta) = \cos 2\theta$

$D = \{(r,\theta) : -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4},$
$0 \leq r \leq \cos 2\theta\}.$

(To find the angles α and β:

$r = 0 = \cos 2\theta$

$\Rightarrow \quad 2\theta = \pi/2 \;{+2k\pi} \quad$ or $\quad 2\theta = -\frac{\pi}{2} \;{+2k\pi}$

$\Rightarrow \quad \theta = \pi/4 \quad$ or $\quad \theta = -\pi/4.$

② Area

$$\text{Area}(D) = \iint_D 1\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} 1 \cdot r\, dr\, d\theta$$

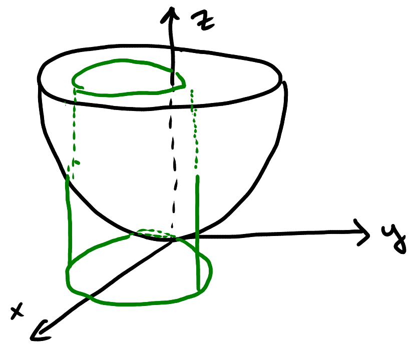
$$= \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\Big|_0^{\cos 2\theta} = \int_{-\pi/4}^{\pi/4} \frac{\cos^2(2\theta)}{2}\, d\theta$$

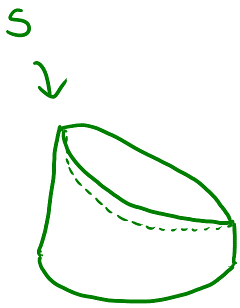
$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos(4\theta)}{2}\, d\theta = \boxed{\frac{\pi}{8}} \approx 0.39$$

EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$. $\rightarrow (x-1)^2 + y^2 = 1$

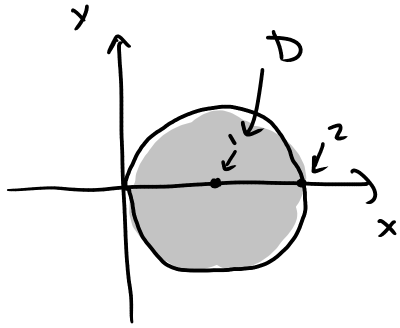
$z \geq 0$

① Picture





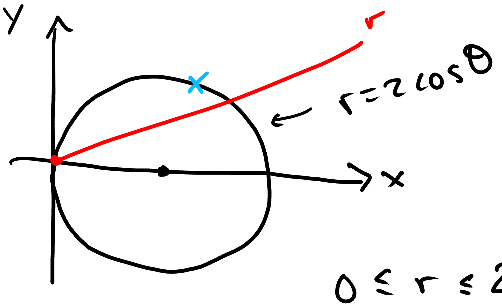
Project it!



Equation of the circle in (r, θ)

$$x^2+y^2 = 2x \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$$
$$\Rightarrow r^2 = 2r\cos\theta$$
$$\Rightarrow r = 2\cos\theta$$

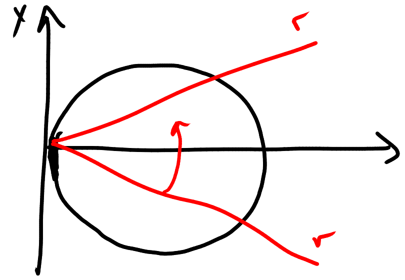
Radius



$0 \leq r \leq 2\cos\theta$

$(1+\cos\theta, \sin\theta)$

Angle



$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$D = \{(r,\theta) : \quad 0 \leq r \leq 2\cos\theta, \ -\tfrac{\pi}{2} \leq \theta \leq \pi/2\}$$

② Volume $\quad z = f(x,y) = x^2+y^2$

$r = \sqrt{x^2+y^2}$

$$\mathrm{Vol}(S) = \iint_D x^2+y^2 \, dA$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \quad r \, dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^3 \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left. \frac{r^4}{4} \right|_{0}^{2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4 \cos^4\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4 \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$\cos^2 2\theta \to \frac{1 + \cos 4\theta}{2}$$

$$= \ldots$$

$$= \boxed{\frac{3\pi}{2}}$$