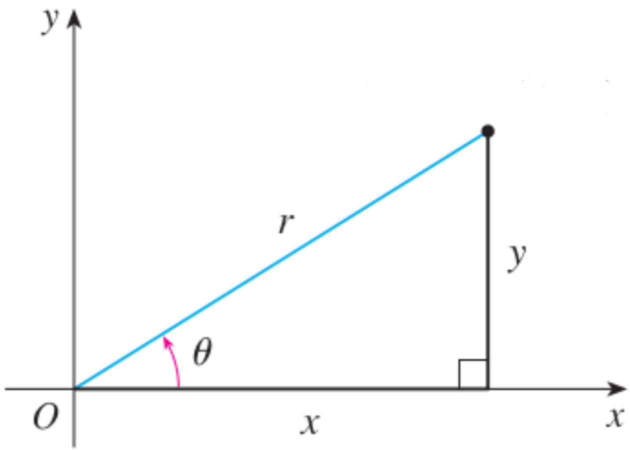
Chapter 15

Multiple Integrals

15.3 Double Integrals in polar coordinates

Polar coordinates

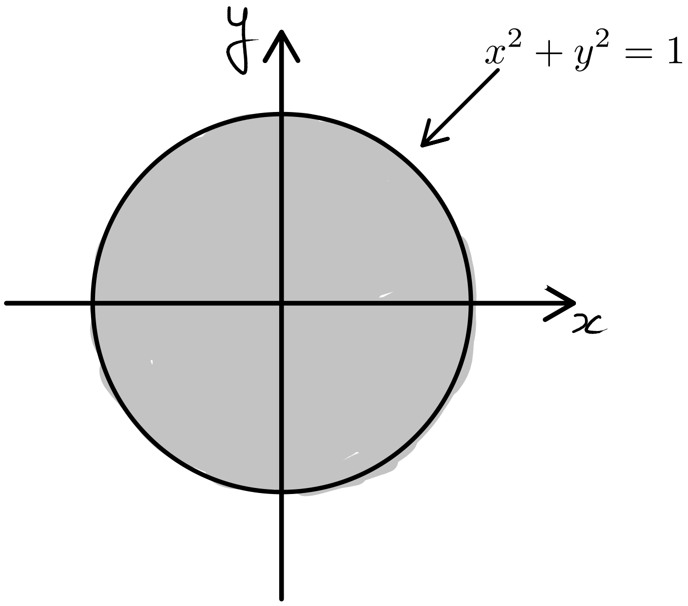


1) Polar to Cartesian:

2) Cartesian to Polar:

Why would we use polar coordinates?

Example. Describe the following region:



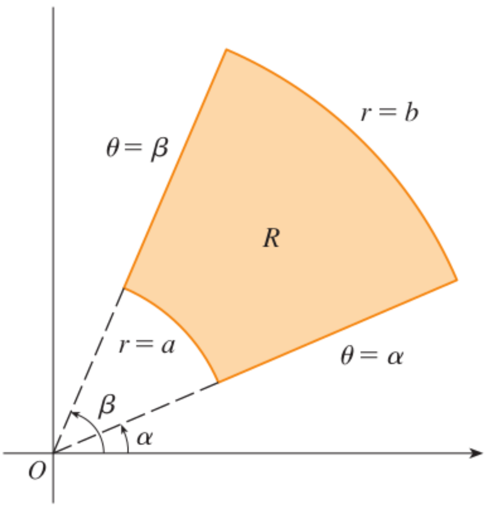
How does it affect the double integral

Recall:

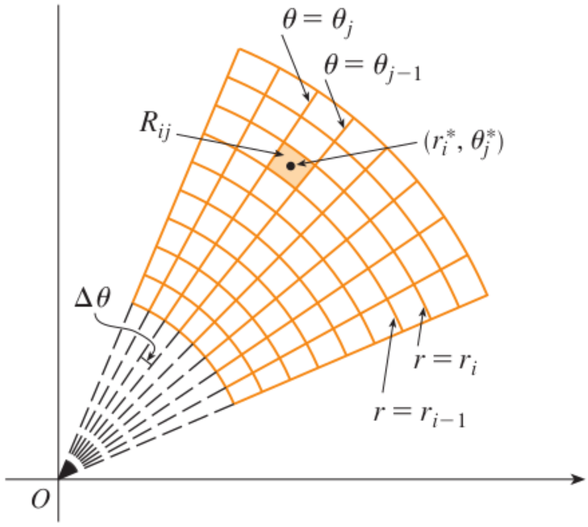
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dxdy \longrightarrow \boxed{dA = dxdy}$$

$$= \int_c^d \int_a^b f(x,y)\,dydx \longrightarrow \boxed{dA = dydx}$$

Polar rectangle:



Divide it $\longrightarrow$



Close-up view

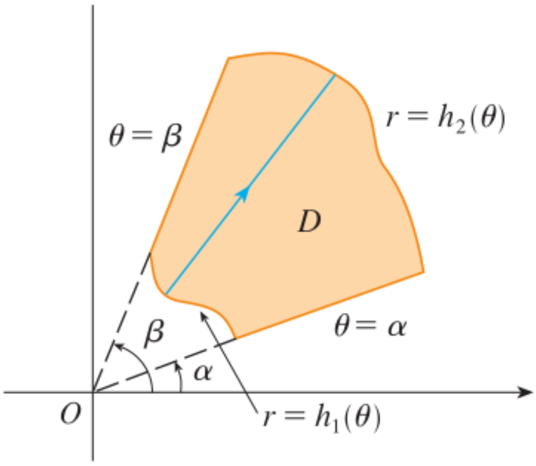
$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,drd\theta}$$

R is a polar rectangle given by $a \le r \le b$ and $\alpha \le \theta \le \beta$, with $\beta - \alpha \le 2\pi$.

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

More complicated region:



3 If f is continuous on a polar region of the form

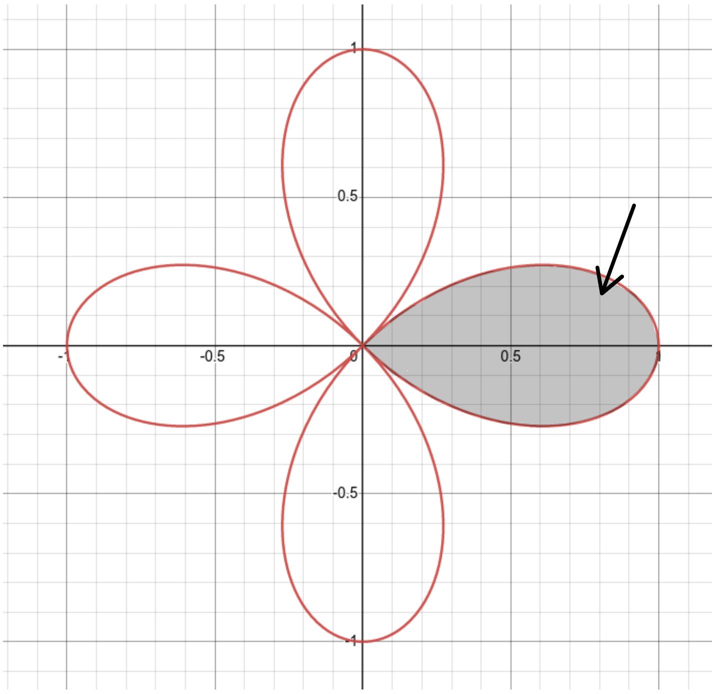
$$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\, r\sin\theta)\, r\,dr\,d\theta$$

EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$\boxed{1}$ PICTURE



EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.