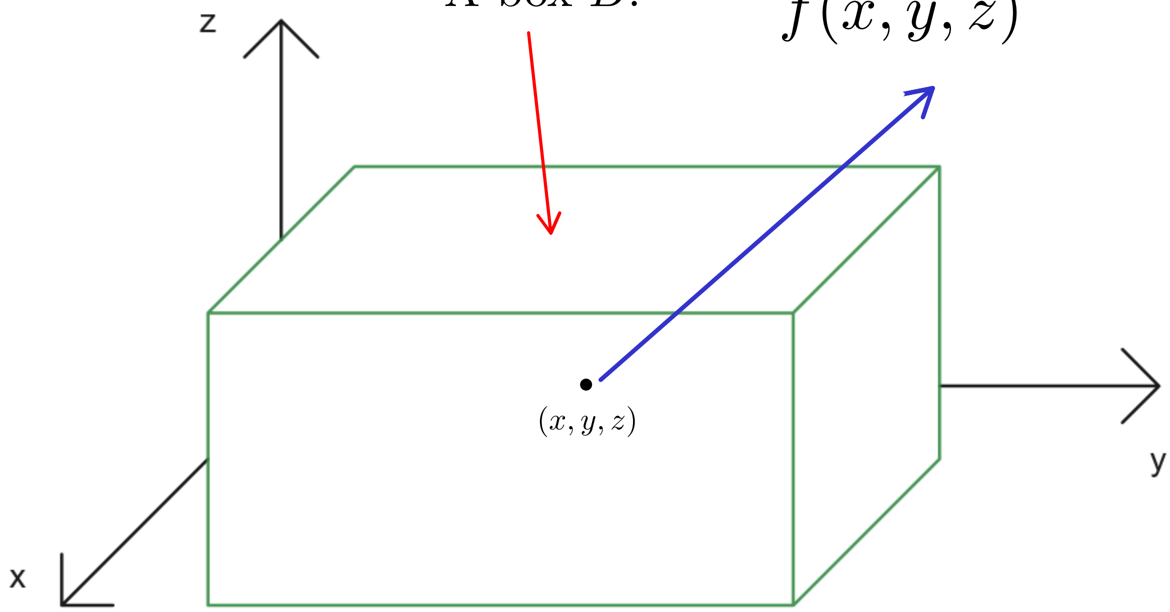
Chapter 15

Multiple Integrals

15.6 Triple integrals

Definition

A box B.



- A box $B = [a, b] \times [c, d] \times [r, s]$
- Divide [a, b] in l parts
- Divide [c, d] in m parts
- Divide [r, s] in n parts

Heat in a small box

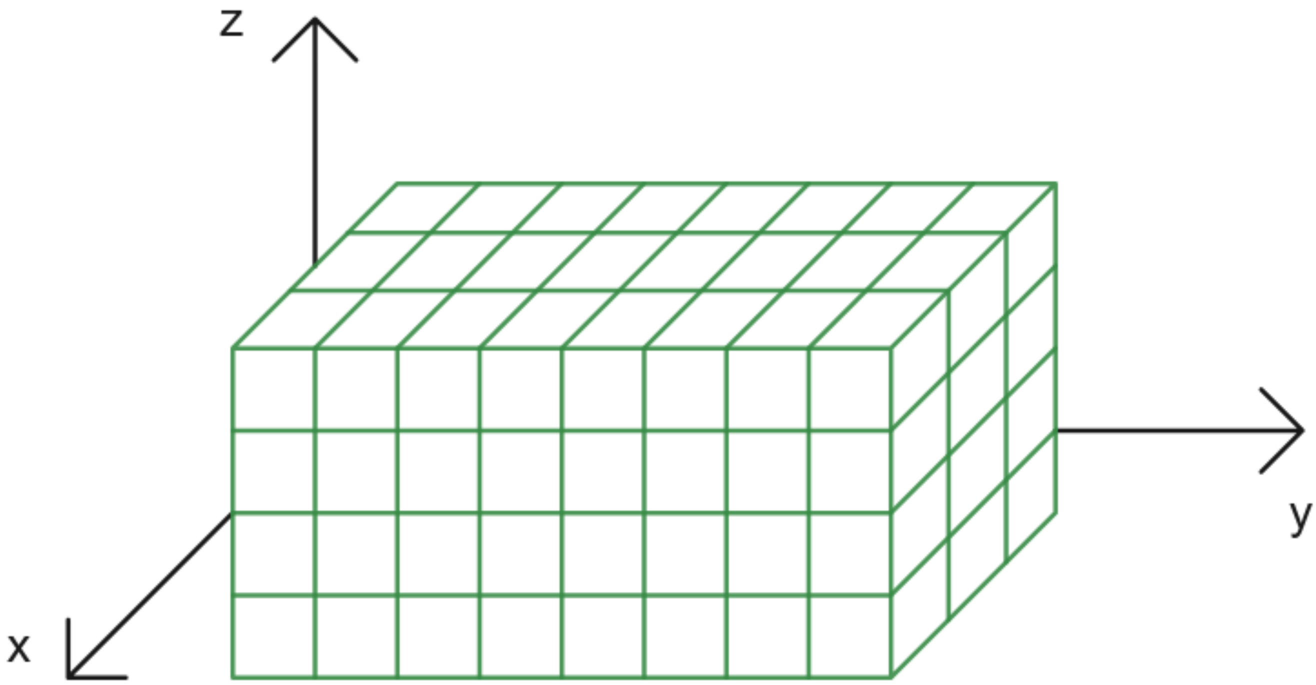
Total heat in box

The triple integral of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

Triple integrals in cartesian coordinates

- Write explicitly $B = \{(x, y, z) : a \le x \le b,\, c \le y \le d,\, r \le z \le s\}$
- Divide $[a, b]$ in parts of length Δx.
- Divide $[c, d]$ in parts of length Δy.
- Divide $[r, s]$ in parts of length Δz.



Fubini's Theorem for triple integrals

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx dy dz$$

the function must be continuous on the box B.

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

QUESTION. What are the 5 other configurations of dx, dy, dz in a triple integral?

1

4

2

5

3

General Domains.

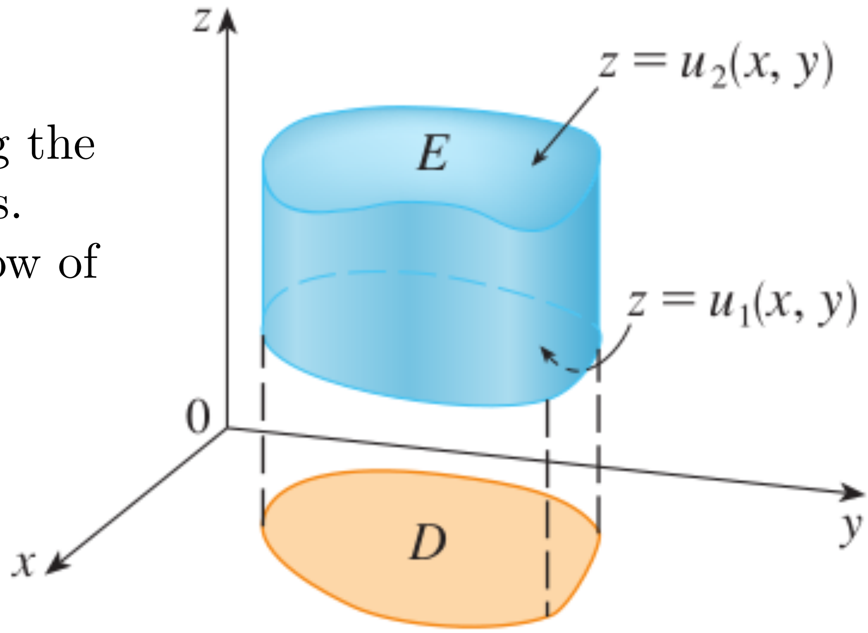
For E a general solid, let B be a box containing E.
Define a function F on B :

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \in B \backslash E. \end{cases}$$

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

Domain of type 1.

- Solid E is bounded along the $z-$axis by two functions.
- Define D to be the shadow of E in the $xy-$plane.
- The domain D can be of type I or type II.

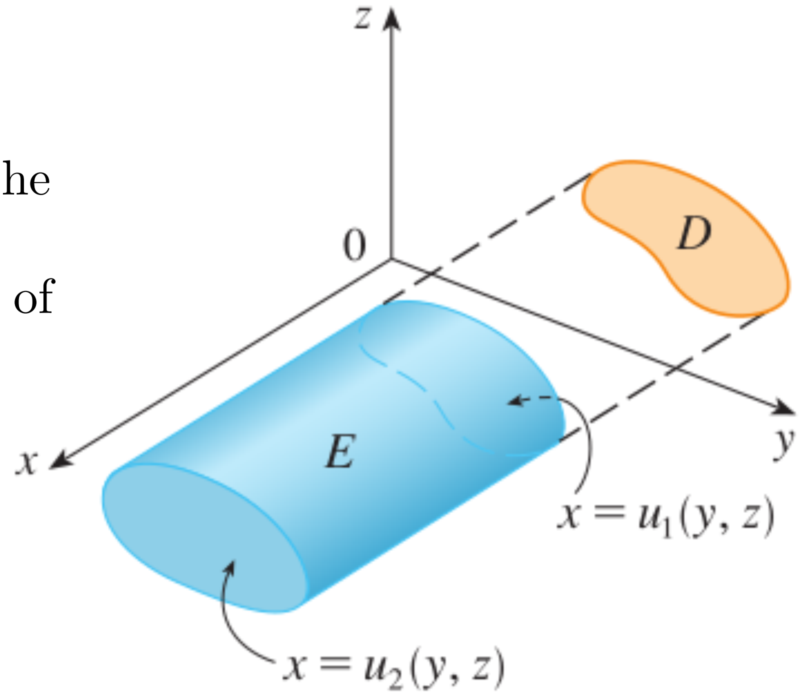


$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

EXAMPLE 2 Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Domains of type 2.

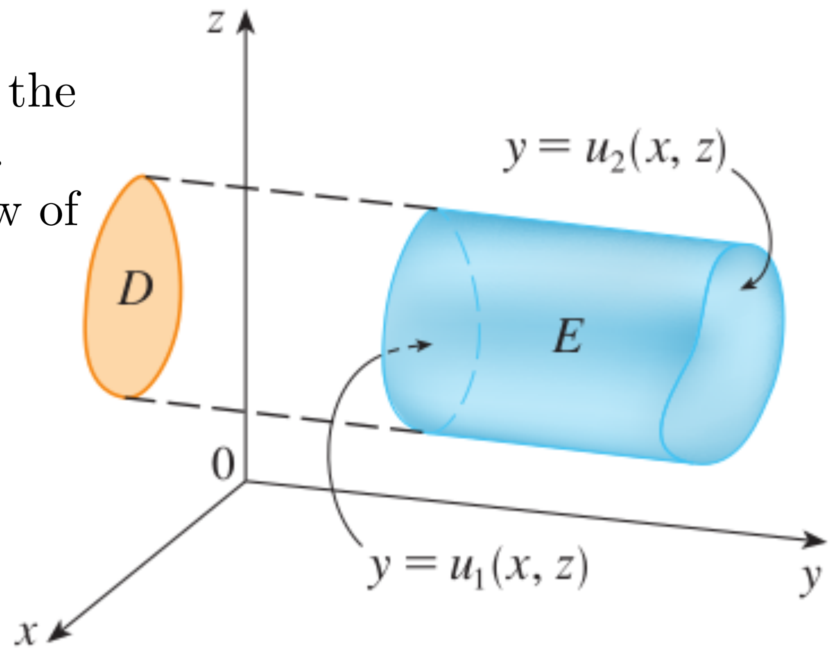
- Solid E is bounded along the $x-$ axis by two functions.
- Define D to be the shadow of E in the $yz-$ plane.
- The domain D can be of type I or type II.



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

Domains of type 3.

- Solid E is bounded along the $y-$ axis by two functions.
- Define D to be the shadow of E in the $xz-$ plane.
- The domain D can be of type I or type II.



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Application: computing volumes of solids.

$$\text{Vol}(E) = \iiint_E dV$$

EXAMPLE. Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2, x = 2y, x = 0,$ and $z = 0.$